Preparation work for A2 Mathematics [2022]

The work studied in Y12 will form the foundations on which will build upon in Year 13. It will only be reviewed during Year 13, it will <u>not</u> be retaught. This is to allow time to revise fully before the exams in June of Y13.

A level Mathematics gets progressively harder with each new topic building on what has been learnt previously. A significant amount of the Year 13 content rests on the work studied throughout year 12.

It is essential that you understand this work and can apply it to exam style questions; otherwise you will struggle to access large parts of the Year 13 course.

This piece of work is designed to review the key topics that you have covered during year 12 and prepare you for the style of questions in the final exams. It consists of two parts, Section A includes notes, examples and exercises to help review and consolidate the skills required to complete Section B. The answers for section A are included to enable you to assess your own understanding. Section B are exam style questions on the topics met throughout section A. This section will be submitted to your class teacher for marking at the start of year 13.

It is to be completed to a high standard with well-structured and clear solutions. You may also need to look things up from your Year 12 notes.

To give yourself the best possible start to Year 13 I would encourage you to devote the time and effort needed to do this work well. You will be required to do corrections to the questions that you don’t get right.

<u>**Section B will be due in at the start of Y13.**</u>

Section A

1. Read through the notes and examples to review the topics.
2. Complete the exercises at the end of each section.
3. Mark your work using the solutions at the end.

1. Transformations of Graphs

- You need to know the basic graphs: $y = x^2$, $y = (x - b)^2 + c$ [ie completed square form], $y = x^3$, $y = {}^1/_x$, y = sin x, y = cos x, y = tan x and the transformation results below.

- Transforming the graph of y = f(x)

  - y = a f (x) or ${}^y/_a$ = f(x) — stretch in the y direction scale factor a
  - y = f (${}^x/_a$) — stretch in the x direction scale factor a

  - y = f (x) + a or y – a = f (x) — translation '+a' in the y direction
  - y = f (x – a) — translation '+a' in the x direction

  - y = – f (x) or – y = f(x) — reflection in the line y = 0 (x axis)
  - y = f(–x) — reflection in the line x = 0 (y axis)

- You need to be able to sketch a transformed basic graph and label its intercepts and/or stationary points

- You also need to be able to describe the transformation that maps a basic graph to a transformed graph or vice versa.

Eg 1 Quadratic Translation

The original graph is $y = x^2$

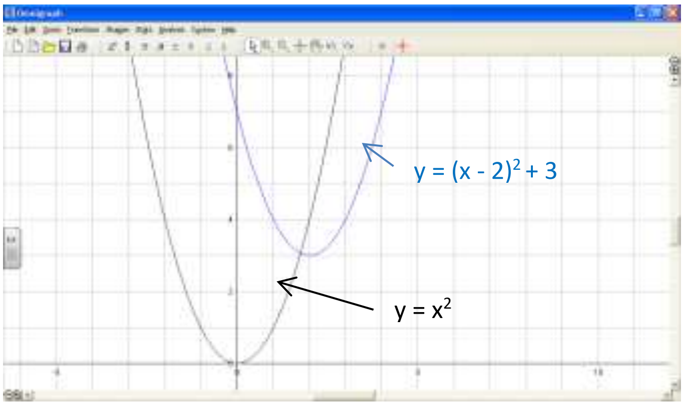
It has been translated through $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Replace x with x – 2 and y with y – 3.

The equation of the transformed graph is

$y - 3 = (x - 2)^2$

or $y = (x - 2)^2 + 3$



Eg 2 Trigonometric Translation

The original graph is y = sin x

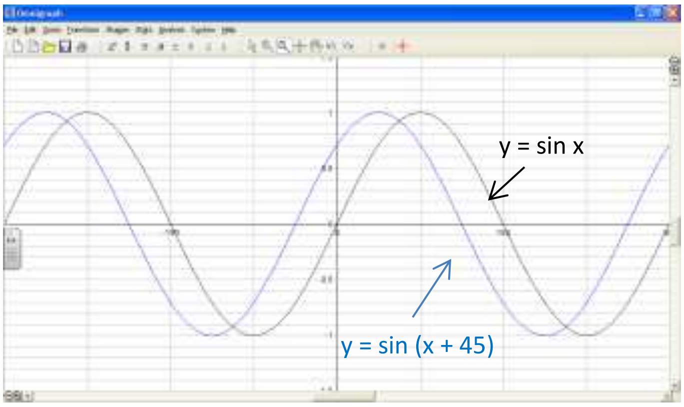
It has been translated through $\begin{bmatrix} -45 \\ 0 \end{bmatrix}$

Replace x with x – (– 45)

The equation of the transformed graph is

y = sin (x – – 45)

y = sin (x + 45)



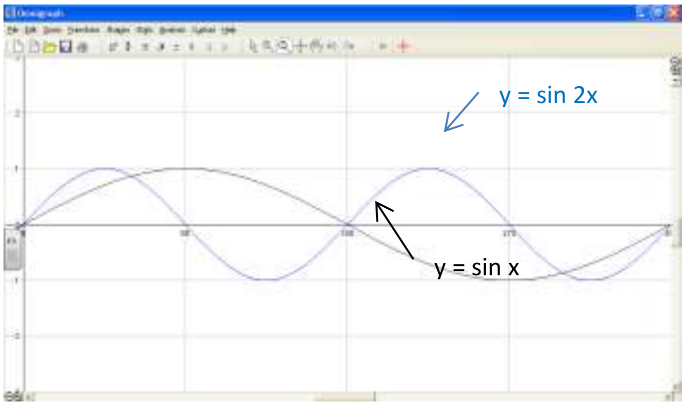
Eg 3 Trigonometric Stretch [x direction]

The original graph is y = sin x
It has been stretched in the x direction
scale factor ½

Replace x with x ÷ ½

The equation of the transformed graph is
$y = \sin(x \div \frac{1}{2})$

$y = \sin 2x$



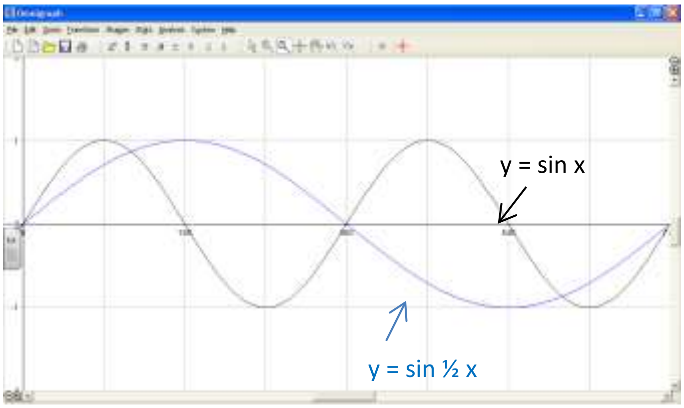
Eg 4 Trigonometric Stretch [x direction]

The original graph is y = sin x
It has been stretched in the x direction
scale factor 2

Replace x with $^x/_2$

The equation of the transformed graph is
$y = \sin \frac{1}{2} x$

[Note in this sketch $0 \le x \le 720$]



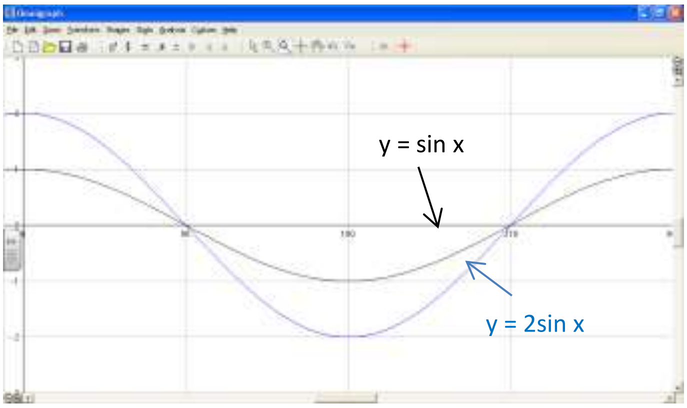
Eg 5 Trigonometric Stretch [y direction]

The original graph is y = sin x
It has been stretched in the y direction
scale factor 2

Replace y with $^y/_2$

The equation of the transformed graph is
$^y/_2 = \sin x$

or $y = 2 \sin x$



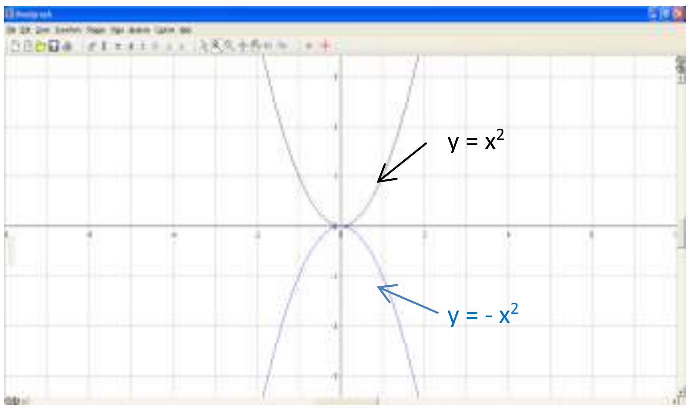
Eg 6 Reflection

The original graph is $y = x^2$
It has been reflected in the x axis [Line y = 0]

Replace y with - y

The equation of the transformed graph is

$-y = x^2$
or $y = -x^2$



- In Pure when there are multiple transformations they will be independent so the order that you do them in does not matter.

  For example y = sin x translated through $\begin{bmatrix} -45 \\ 0 \end{bmatrix}$ and stretched sf 2 in the y direction

  Doing the translation then the stretch gives: first $y = \sin(x + 45)$
  then $^{y}/_{2} = \sin(x + 45)$ or $y = 2\sin(x + 45)$

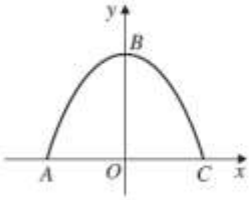
  Doing the stretch then the translation gives: first $^{y}/_{2} = \sin x$ or $y = 2\sin x$
  then $y = 2\sin(x + 45)$

However it is not always the case that transformations are independent and this will be looked at in Core 3.

1 Describe the transformation that maps $y = x^2$ onto:

a $y = (x - 2)^2$ **b** $y = 3x^2$

2 The diagram shows a sketch of $y = f(x)$.
A is the point $(-1, 0)$, B is $(0, 1)$ and C is $(1, 0)$.



State the coordinates of the new positions of A, B and C under these transformations:

a Stretch in the y-direction, factor 2.

b Reflection in the x-axis.

c Stretch in the x direction, factor 0.5.

3 **a** Describe the transformation that maps $y = x^2$ onto $y = x^2 - 4x + 7$.

b Describe the transformation that maps $y = x^2 + 8x + 6$ onto $y = x^2$.

Hint:
Complete the square.

4 Sketch the following:

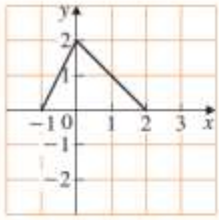
a $y = \dfrac{1}{x}$ **b** $y = \dfrac{1}{x} + 2$ **c** $y = \dfrac{1}{x+2}$.

5 Describe the transformation that maps $y = (x + 3)^2$ onto $y = (x - 1)^2$.

6 It is given that $f(x) = x^2$. Under which of the following transformations does the graph of $y = f(x)$ remain unchanged?

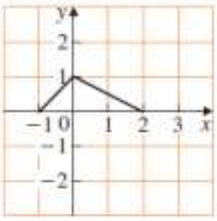
a $y = f(-x)$ **b** $y = -f(x)$ **c** $y = f(x - 1)$

7 The diagram on the right shows the graph of $y = f(x)$.



Describe each of the following graphs in terms of $y = f(x)$.

a



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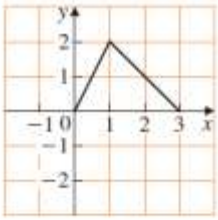
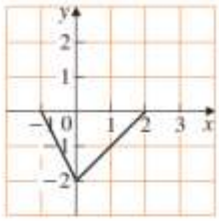
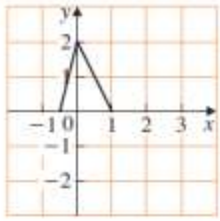


c



d



8 Sketch the graphs of the following circles:

a $(x - 1)^2 + y^2 = 4$ **b** $x^2 + (y + 3)^2 = 25$

c $(x - 1)^2 + (y + 2)^2 = 9$ **d** $(x + 3)^2 + (y - 2)^2 = 4$

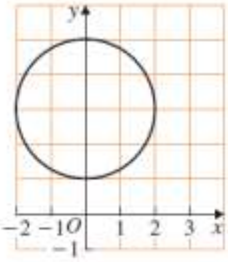
9 For the graphs in question **8**,

a state the translation of the circle $x^2 + y^2 = a^2$ that they each represent

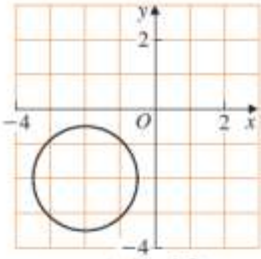
b state the centre and radius of each circle.

10 State the equations of each of the following graphs

a



b



(radius 1.5)

11 Sketch the graphs of the following curves showing the vertex, and state the translation of $y = x^2$ that they each represent.

a $y = (x + 4)^2 - 1$ **b** $y = (x + 3)^2 + 2$ **c** $y = (x - 2)^2 + 3$

12 **a** Describe the transformation that maps $y = x^3$ onto

i $y = (x - 1)^3$ **ii** $y = x^3 + 1$

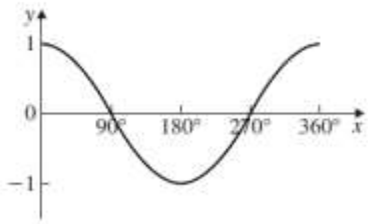
b Sketch the 3 graphs on the same axes.

13 Two transformations map $y = x^2$ onto $y = 16x^2$.

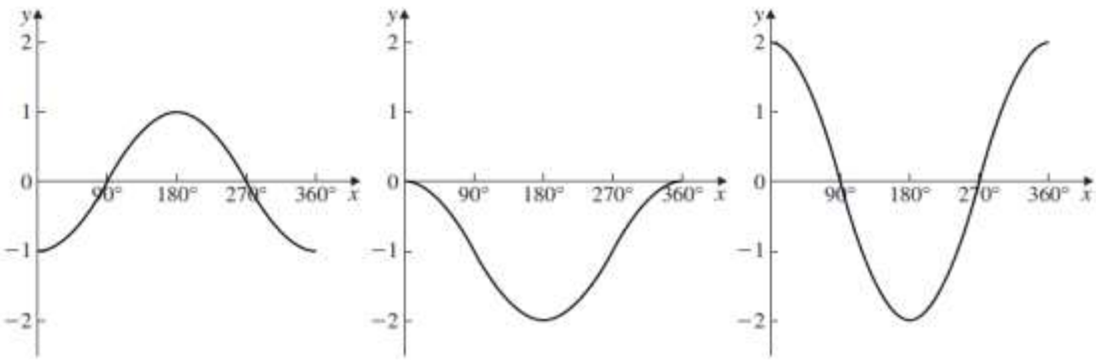
a The first is a stretch in the x-direction. What is the scale factor?

b The second is a stretch in the y-direction. What is the scale factor?

14 The graph shows $y = \cos x$ for $0 \leqslant x \leqslant 360°$



Describe each of the following as transformations of $y = \cos x$ and write down the equation of each graph.



2. Trigonometry [Equations]

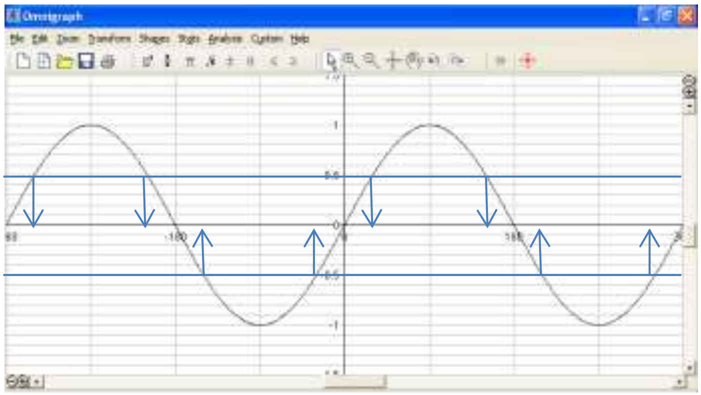
- The trigonometry of any angle can be related to the trigonometry of acute angles using the trigonometric graphs.

The graph of y = sin x is only shown for -360 to 360 but is defined for all values of x.

You can see from the diagram all of the angles whose trigonometry relates to that of 30^0 including: sin (150) = sin (30) = ½
sin (–30) = – sin (30) = – ½

Solving the equation sin x = ½ for -360 < x < 360 gives multiple solutions:

$x = 30^0$, $x = -180 - 30 = -210^0$
$x = 180 - 30 = 150^0$ $x = -360 + 30 = -330^0$

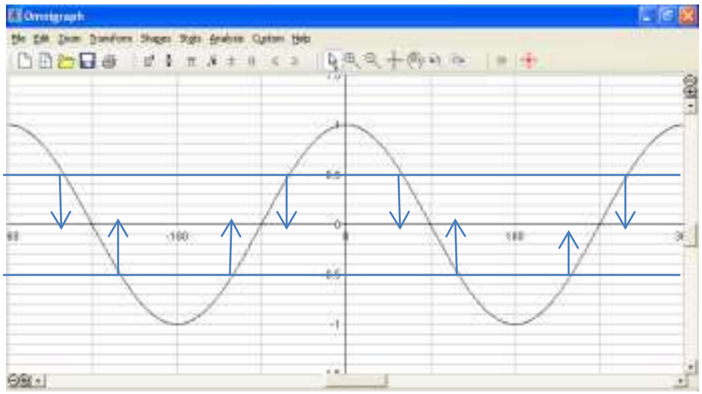


The graph of y = cos x is only shown for -360 to 360 but is defined for all values of x.

You can see from the diagram all of the angles whose trigonometry relates to that of 60^0 including: cos (300) = cos (60) = ½
cos (–120) = – cos (60) = – ½

Solving the equation cos x = ½ for -360 < x < 360 gives multiple solutions:

$x = 60^0$, $x = -60^0$
$x = 360 - 60 = 300^0$ $x = -360 + 60 = -300^0$

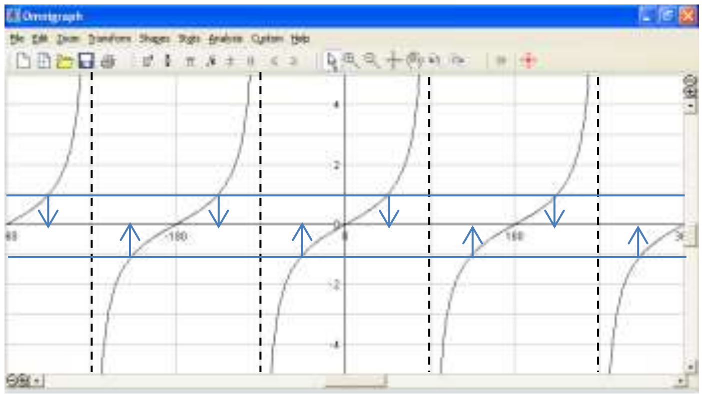


The graph of y = tan x is only shown for -360 to 360 but is defined for all values of x except for where cos x = 0.

You can see from the diagram all of the angles whose trigonometry relates to that of 45^0 including: tan (225) = tan (45) = 1
tan (–45) = – tan (45) = – 1

Solving the equation tan x = 1 for -360 < x < 360 gives multiple solutions:

$x = 45^0$, $x = 45 - 180 = -135^0$
$x = 45 + 180 = 225^0$ $x = -135 - 180 = -315^0$



Basic Trigonometric Equations

- When solving trigonometric equations for a given interval the principal solution is found using your calculator and the other solutions are found using the trigonometric graphs as demonstrated in the work above.

Eg 1 Solve sin x = ½ for 0 < x < 360

$x = \sin^{-1}(½) = 30^0$ [found by using your calculator]
The other solution in the interval is found by looking at the shape of the graph: $x = 180 - 30 = 150^0$

Trigonometric Identities

- You need to learn the following results
  - (i) $\tan x = \sin x / \cos x$
  - (ii) $\sin^2 x + \cos^2 x = 1$ [note: $\sin^2 x$ means $(\sin x)^2$ to avoid confusion with $\sin x^2$ which means $\sin (x^2)$]
    and hence: $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$

- Also learn these techniques to use when trying to prove identities:

  - ✓ Start with the 'most difficult' side
  - ✓ To turn a sum into a product look to pull out a common factor
    eg $\sin^3 x + \sin x \cos^2 x = \sin x (\sin^2 x + \cos^2 x)$
  - ✓ Look out for difference of two squares $1 - \cos^2 x = (1 + \cos x)(1 - \cos x)$
  - ✓ Look out for one of the brackets of a difference of two squares in a fraction such as $(1 + \cos x)$ and create a DOTS by multiplying numerator and denominator by the other bracket, $(1 - \cos x)$
  - ✓ To split a single fraction into a sum of terms – write each term in the numerator over the denominator
    Eg $\frac{sinx+cosx}{cosx} = \frac{sinx}{cosx} + \frac{cosx}{cosx} = tanx + 1$
  - ✓ To turn a sum of terms into a single fraction – write each term over a common denominator
    Eg $1 + \frac{\cos^2 x}{\sin^2 x} = \frac{sin^2 x}{\sin^2 x} + \frac{cos^2 x}{sin^2 x} = \frac{sin^2 x + cos^2 x}{\sin^2 x} = \frac{1}{sin^2 x}$
  - ✓ You can use $1 = \sin^2 x + \cos^2 x$ to replace numbers with trigonometry
  - ✓ Multiply numerator and denominator by the same term to eliminate fractions within fractions

Eg 2 Show that $\frac{\sin^2 x}{1 - \cos x} \equiv 1 + \cos x$

Start with one side and keep manipulating until you get the other side. Do not try to 'do the same to both sides' like when solving equations.

LHS $\equiv \frac{\sin^2 x}{1 - \cos x}$ [Use $\sin^2 x + \cos^2 x = 1$ rearranged to replace $\sin^2 x$]

$\equiv \frac{1 - \cos^2 x}{1 - \cos x}$ [Factorise numerator as DOTS]

$\equiv \frac{(1 + \cos x)\cancel{(1 - \cos x)}}{\cancel{1 - \cos x}}$ [Cancel common factor from numerator and denominator]

$\equiv 1 + \cos x \equiv$ RHS

Eg 3 Show that $\frac{(\sin x + \cos x)^2 - 1}{2\cos^2 x} \equiv \tan x$

LHS $\equiv \frac{\sin^2 x + 2 \sin x \cos x + \cos^2 x - 1}{2\cos^2 x}$ [Use $\sin^2 x + \cos^2 x = 1$]

$\equiv \frac{1 + 2 \sin x \cos x - 1}{2 \cos^2 x}$ [Simplify the numerator by cancelling the 1s]

$\equiv \frac{2 \sin x \cos x}{2 \cos^2 x}$ [Cancel common factors from numerator and denominator]

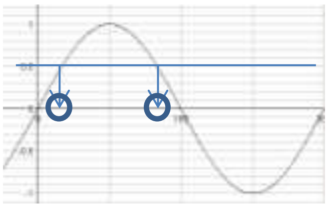
$\equiv \frac{\sin x}{\cos x}$ [Use $\tan x = \sin x / \cos x$]

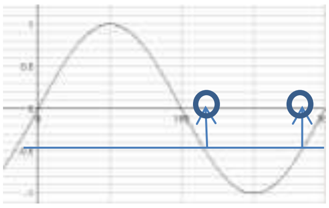
$\equiv \tan x \equiv$ RHS

Quadratic Trigonometric Equations

- The first solution to an equation is found using your calculator and the other solutions in the given interval are found by looking at the shape of the graph. Quadratic Equations may involve difference of two square and common factors as well as standard factorisation.

Eg 4 **Solve $4\sin^2 x - 1 = 0$ for $0 \le x < 360^0$**

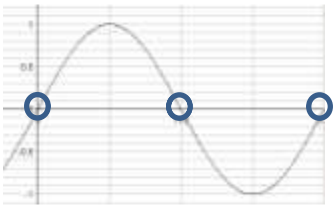


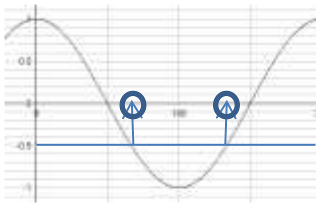


Factorise by difference of two squares:
$4\sin^2 x - 1 = 0$
$(2\sin x - 1)(2\sin x + 1) = 0$
$\sin x = ½$, $\sin x = -½$

sin x = ½ gives: $x = \sin^{-1}(½) = 30^0$ and $x = 180 - 30 = 150^0$
sin x = - ½ gives: $x = \sin^{-1}(-½) = -30^0$ which is outside the interval
so $x = 180 + 30 = 210^0$ and $x = 360 - 30 = 330^0$

Eg 5 **Solve $2\sin x \cos x + \sin x = 0$ for $0 < x < 360^0$**





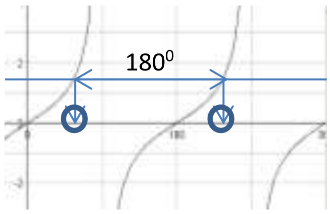
Do NOT divide by sin x as this causes you to lose solutions!

Instead factorise by common factor:
$2\sin x \cos x + \sin x = 0$ $\quad \sin x(2\cos x + 1) = 0$
$\sin x = 0$, $\cos x = -½$

sin x = 0 gives: $x = 180^0$ [Note 0^0 and 360^0 are NOT in the interval]
cos x = - ½ gives: $x = \cos^{-1}(-½) = 120^0$ and $x = 360 - 120 = 240^0$

- To solve an equation involving two different trigonometric functions which CANNOT be factorised by common factor [like in example 5] you often need to use trigonometric identities…

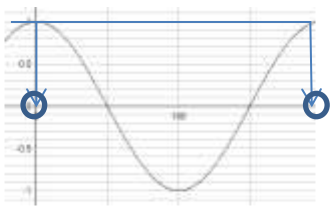
Eg 6 **Solve $2\sin x - 3\cos x = 0$ for $0 < x < 360^0$**
Give your answers to 3sf.



Rearrange and use the identity tan x = sin x / cos x:
$2\sin x - 3\cos x = 0$
$2\sin x = 3\cos x$
$\frac{\sin x}{\cos x} = \frac{3}{2}$, hence: $\tan x = 3/2$

$x = \tan^{-1}(^3/_2)$ gives: $x = 56.3^0$ (3sf) and $x = 180 + 56.3099… = 236^0$ (3sf)

Eg 7 **Solve $\sin^2 x = 2(\cos x - 1)$ for $0 \le x \le 360^0$**



Substitute $\sin^2 x = 1 - \cos^2 x$ to create a quadratic equation in cos x:
$1 - \cos^2 x = 2\cos x - 2$
$\cos^2 x + 2\cos x - 3 = 0$
$(\cos x - 1)(\cos x + 3) = 0$
$\cos x = 1$, $\cos x = -3$

cos x = 1 has solutions x = 0 and x = 360 [from the graph - note they ARE in the interval for this question].
cos x = – 3 has no solutions as $-1 \le \cos x \le 1$ for all x.

Trigonometric Equations involving a substitution and a change of interval

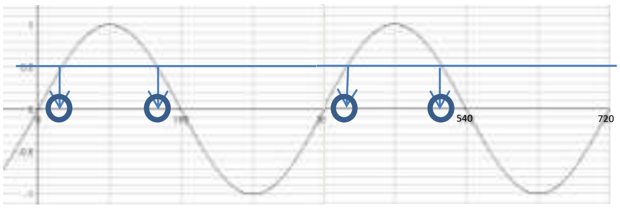
- These equations contain a multiple of x ie $\sin 3x = ½$ or $\sin(2x - 20) = 0.5$

- A basic equation like $\sin x = ½$ has two solutions in the interval is $0 \le x \le 360$. A question involving 2x usually has 4 solutions in this interval. A question involving 3x usually has 6 solutions in this interval.

- Learn the strategy for solving these type of questions: $\sin(2x - 20) = 0.5$ for $0 \le x \le 360^0$

  (i) Let $u = 2x - 20$ so the question becomes $\sin u = 0.5$
  (ii) The interval for x is $0 \le x \le 360^0$ so the interval for u becomes $2(0) - 20 \le u \le 2(360) - 20^0$
  Ie $-20 \le u \le 700^0$
  (iii) Draw the trig graph $y = \sin u$ for $-20 \le u \le 700^0$
  (iv) Solve $\sin u = 0.5$ using your calculator and find all of the other solutions for u from your graph
  (v) $u = 2x - 20$ so $x = (u + 20)/2$
  Use the substitution to change all of your solutions for u into solutions for x

Eg 8 Solve $\sin(2x - 20) = 0.5$
for $0 \le x \le 360^0$



Let $u = 2x - 20$
Hence $\sin u = 0.5$
for $-20 \le u \le 700^0$

$\sin u = 0.5$,
$u = \sin^{-1}(0.5) = 30^0$

From the graph there are no solutions for u in the interval -20 to 0 but there are 3 more solutions in the interval 0 to 700:
$u = 180 - 30 = 150^0$
$u = 360 + 30 = 390^0$
$u = 540 - 30 = 510^0$

Using the substitution: $u = 2x - 20$ we have $x = (u + 20)/2$

Hence: $u = 30$ $x = (30 + 20) / 2 = 25^o$
$u = 150$ $x = (150 + 20) / 2 = 85^o$
$u = 390$ $x = (390 + 20) / 2 = 205^o$
$u = 510$ $x = (510 + 20) / 2 = 265^o$

Final answers: $x = 25^o, 85^o, 205^0, 265^o$

- Note: if your answers for u are not whole numbers you should not round them as potentially this can give you slightly incorrect answers for x.

SKILLS CHECK 1C EXTRA: Trigonometric functions, identities and equations

1 Solve the following equations for $0° \leqslant x \leqslant 360°$. If your answer is not exact, give it correct to the nearest degree.

a $\cos x = 0.4$ **b** $\sin^2 x = 0.25$ **c** $\tan(x + 30°) = 1.5$

2 Solve the following equations for $0 \leqslant x \leqslant 180°$

a $\cos 2x = -\frac{\sqrt{3}}{2}$ **b** $\tan 3x = \frac{1}{\sqrt{3}}$ **c** $\sin(0.5x) = 0.5$

3 Find all the values of x in the interval $0 < x < 360°$ for which $\sin x = -0.15$, giving your answers in degrees correct to two decimal places.

4 Find all the values of x in the interval $0° \leqslant x \leqslant 360°$ for which $2\sin^2 x = \sqrt{3}\sin x$.

5 Show that $\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$, for all values of θ.

6 Show that $\tan^2 x(1 + \sin x) = \frac{\sin^2 x}{1 - \sin x}$, for all values of x.

7 It is given that $\cos^2 30° - \sin 45° = a + b\sqrt{2}$, where a and b are rational numbers. Find a and b.

8 Find the exact value of $\sin\theta$, given that $\tan\theta = \frac{5}{12}$ and $\cos\theta = -\frac{12}{13}$.

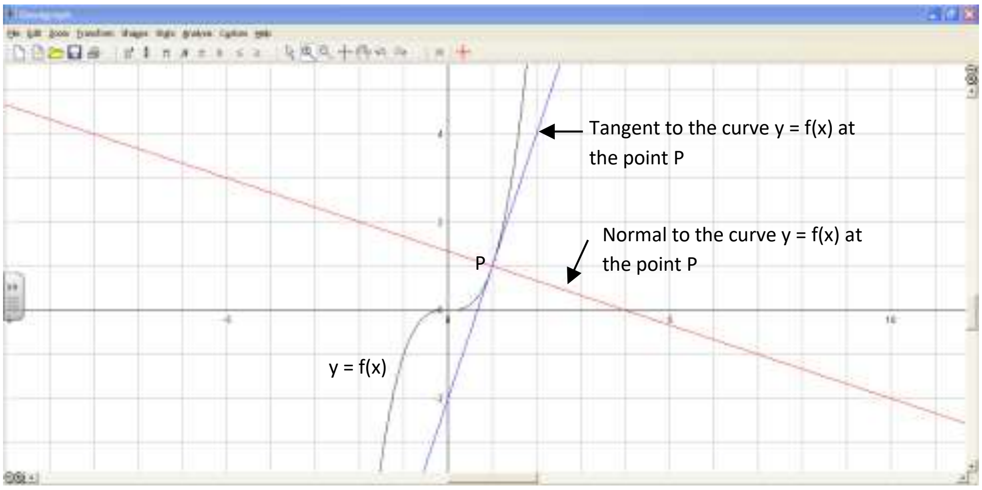
9 Find all the values of x in the interval $0 \leqslant x \leqslant 360°$ for which $\cos\left(x - 60°\right) = 0.5$.

10 **a** Given that $3\sin^2 x + \cos^2 x - \cos x = 2$, show that $2\cos^2 x + \cos x - 1 = 0$.

b Hence find all the solutions of the equation $3\sin^2 x + \cos^2 x - \cos x = 2$ in the interval $0° \leqslant x \leqslant 360°$.

3. Differentiation

Calculating Gradients



- The gradient of a curve at a point P on the curve is the gradient of the tangent to the curve at that point. Differentiation is the process of calculating the gradient function which is a rule to find the gradient at any point on the curve. The notation for gradient is $^{dy}/_{dx}$ or f'(x).

- The rules for differentiating are: For $y = kx^n$ $\frac{dy}{dx} = nkx^{n-1}$ for n being any rational number. Any constant differentiates to 0.

- The gradient at a specific point is found by substituting the x coordinate of the point into the gradient function.

- You always **differentiate term by term** so you must expand any brackets or break up any fractions [by writing each term over the denominator] prior to differentiating

Eg 1 y = $\frac{(\sqrt{x}-4)^2}{2\sqrt{x}}$ **Find the gradient of the curve at the point where x = 4.**

y = $\frac{(\sqrt{x}-4)^2}{2\sqrt{x}} = \frac{x-8\sqrt{x}+16}{2x^{1/2}} = \frac{1}{2}x^{1/2} - 4 + 8x^{-1/2}$

$\frac{dy}{dx} = \frac{1}{4}x^{-1/2} + 0 - 4x^{-3/2} = \frac{1}{4\sqrt{x}} - \frac{4}{(\sqrt{x})^3}$

When x = 4 $^{dy}/_{dx} = \frac{1}{8} - \frac{4}{8} = \frac{-3}{8}$

- You could be given a function and the value of the gradient at a specific point and work back towards the points on the curve.

Eg 2 Find the coordinates of the point on the curve y = $\frac{1}{x^2} + 5$ where the gradient = - ¼

y = x^{-2} + 5 $^{dy}/_{dx}$ = $-2x^{-3}$ Hence $-2x^{-3}$ = - ¼

$\frac{-2}{x^3} = \frac{-1}{4}$

8 = x^3

2 = x

When x = 2 y = $\frac{1}{x^2}$ + 5 = ¼ + 5. Hence the required coordinates are (2, 5¼)

Tangents and Normals

- If P = (a, b) is a point on the curve y = f(x) then the gradient of the tangent at point P is the same as the gradient of the curve at P i.e. f′(a).
  The equation of the tangent to the curve at the point P is… y – b = f′(a) (x – a)

- The normal at P is perpendicular to the tangent so its gradient = $^{-1}/_{f'(a)}$
  The equation of the normal to the curve at the point P is … y – b = $^{-1}/_{f'(a)}$ (x – a)

Eg 3 Find the equation of the normal to the curve y = $\frac{(\sqrt{x}-4)^2}{2\sqrt{x}}$ at the point where x = 4.

When x = 4, y = $\frac{(2-4)^2}{4}$ = 1 so (a, b) = (4, 1)

From eg 1: the gradient of the curve y = = $\frac{(\sqrt{x}-4)^2}{2\sqrt{x}}$ at x = 4 was – $^3/_8$

Hence the gradient of the normal = $^8/_3$

The equation of the normal is: y – 1 = $^8/_3$ (x – 4)

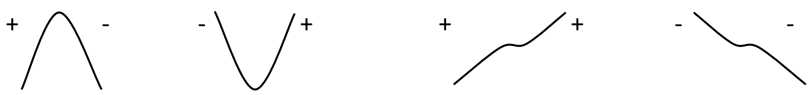
Stationary Points

- Stationary [or turning] points occur where f ′(x) = 0. To find the x coordinates of stationary points, solve the equation f ′(x) = 0. You can find the corresponding y coordinates using the equation of the curve y = f(x).

- A stationary point may be a [local] maximum, a [local] minimum or a point of inflection. The nature of a stationary point can be determined by using the second derivative $\frac{d^2y}{dx^2}$ of f′′(x)

At stationary point (a,b) ie where x = a If f ′′(a) > 0 the point is a minimum
If f ′′(a) < 0 the point is a maximum
If f ′′(a) = 0 then abandon this method and use the method below instead

- The nature of a stationary point can also be determined by considering the value of the gradient , $^{dy}/_{dx}$, either side of the stationary point



MAX MIN POINTS OF INFLECTION

Eg 7 f(x) = $\frac{2(x+3)}{\sqrt{x}}$. Find the coordinates of the stationary point and determine its nature.

f(x) = $2x^{1/2} + 6x^{-1/2}$

f′(x) = $x^{-1/2} - 3x^{-3/2} = \frac{1}{\sqrt{x}} - \frac{3}{x\sqrt{x}}$

At stationary points f′(x) = 0 hence $\frac{1}{\sqrt{x}} - \frac{3}{x\sqrt{x}}$ = 0

Multiplying through by x√x gives: x – 3 = 0 Hence x = 3 is the x coordinate of the stationary point

When x = 3, y = $\frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

The coordinates of the stationary point are (3, 4√3)

f′′(x) = $\frac{-1}{2}x^{-3/2} + \frac{9}{2}x^{-5/2} = \frac{-1}{2x\sqrt{x}} + \frac{9}{2x^2\sqrt{x}}$

When x = 3 f′′(x) = $\frac{-1}{6\sqrt{3}} + \frac{9}{18\sqrt{3}} = \frac{-1}{6\sqrt{3}} + \frac{3}{6\sqrt{3}} = \frac{1}{3\sqrt{3}}$ > 0 Hence (3, 4√3) is a minimum point

SKILLS CHECK **4B EXTRA:** Applications of differentiation

1 Find the equation of the tangent to the curve $y = (2x - 1)^2$ at the point where $x = -1$, giving your answer in the form $y = mx + c$.

2 Find the equation of the normal to the curve $y = (2x - 1)(2x + 1)$ at the point where $x = 1$. Give your answer in the form $ax + by + c = 0$.

3 The curve $y = 1 - x^3$ crosses the x-axis at the point A.

a Find the coordinates of A.

b Find the gradient of the normal at A.

c Find the equation of the normal at A.

d Find the coordinates of B, the point where the normal at A meets the y-axis.

e Find the area of triangle AOB, where O is the origin.

4 Find the equation of the tangent to the curve $y = 3x^2(1 - x)$ at $x = 1$ and show that this is parallel to the line $y + 3x - 4 = 0$.

5 Find the set of values of x for which $f(x)$ is decreasing, where $f(x) = x^2 - 9x$.

6 Find the set of values of x for which $f(x)$ is increasing, where $f(x) = 3x^2 - 2x + 1$.

7 Find the equation of the tangent to the curve $y = \dfrac{x^2 - 1}{\sqrt{x}}$ at the point where $x = 4$, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers.

8 The point P, with x-coordinate -1, lies on the curve $y = 2x^2 + 4 - \dfrac{1}{x^2}$.

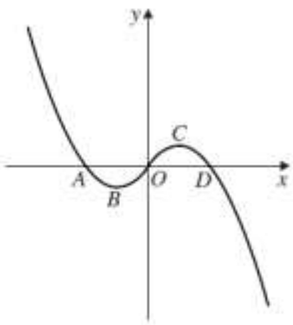
a Find the gradient of the tangent to the curve at P.

b Find the equation of the normal to the curve at P.

SKILLS CHECK **4C EXTRA:** Stationary points and problems

1 Find the stationary point of the curve $y = 6x - x^2$, stating its nature.

2 Below is a sketch of the curve $y = 4x - x^3$.



Find the x-coordinates of the points A, B, C and D.

3 Find the values of a and b if the curve $y = ax^2 - 8x + b$ crosses the y-axis at the point $(0, 3)$ and has a stationary point when $x = 2$.

4 Find the greatest value of $f(x)$, where $f(x) = 16x - 4x^2 - 1$.

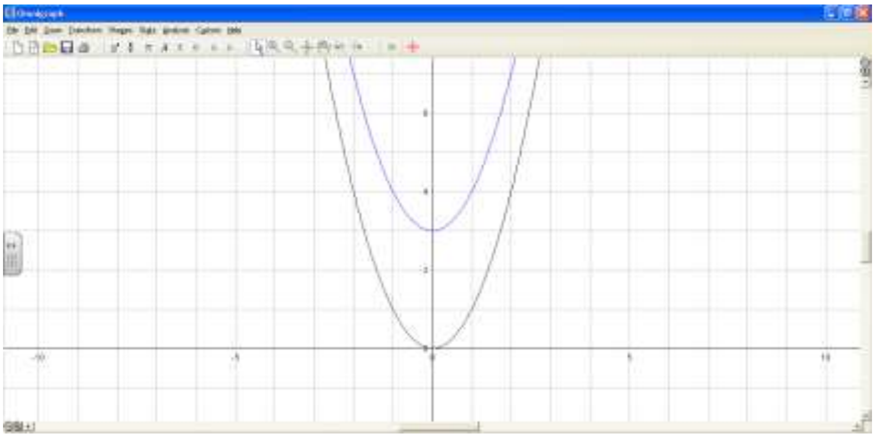
5 Find the equation of the tangent to the curve $y = 2x^2 - 8x + 1$ at the point where the curve has a stationary point.

6 Find the coordinates of the turning points of the curve $y = 27x - x^3 - 1$, stating whether the turning point is a maximum or a minimum.

4. Integration

Introducing Integration

- Integration is the reverse of differentiation – it can be used to find a function from its gradient function.



The diagram shows the curves y = x^2 and y = $x^2 + 3$. These functions both have the same gradient function f′(x) = 2x. There are infinitely many curves with gradient function = 2x but they are all vertical translations of the graph of y = x^2. This group of functions can be described as y = x^2 + c where c is a constant.

When integrating we need to indicate +c to show all of the possible functions. If we know a point on the curve then we can find the value of c and hence one specific function.

If dy/dx = f′(x) then we say y = $\int f'(x)dx = f(x)+c$ where c is the constant of integration.

- To integrate: If $y = kx^n$ then $\int y \ dx = \dfrac{kx^{n+1}}{n+1} + c$ for n being any rational number except -1.

- You always **integrate term by term** so you must expand any brackets or break up any fractions [by writing each term over the denominator] prior to integrating

Eg 1 If f′(x) = √x (5x – 6) and (2, 3) lies on the curve find the equation of the curve y = f(x)

y = $\int \sqrt{x}(5x-6)\ dx$

y = $\int 5x^{3/2} - 6x^{1/2}\ dx$

y = $\dfrac{5x^{5/2}}{5/2} - \dfrac{6x^{3/2}}{3/2} + c$

y = $2x^{5/2} - 4x^{3/2} + c$

When x = 2, y = 3 so... 3 = $2\left(\sqrt{2}\right)^5 - 4\left(\sqrt{2}\right)^3 + c$

3 = $2 \times 4\sqrt{2} - 4 \times 2\sqrt{2} + c$

3 = $8\sqrt{2} - 8\sqrt{2} + c$ hence c = 3

Hence y = $2x^{5/2} - 4x^{3/2} + 3$

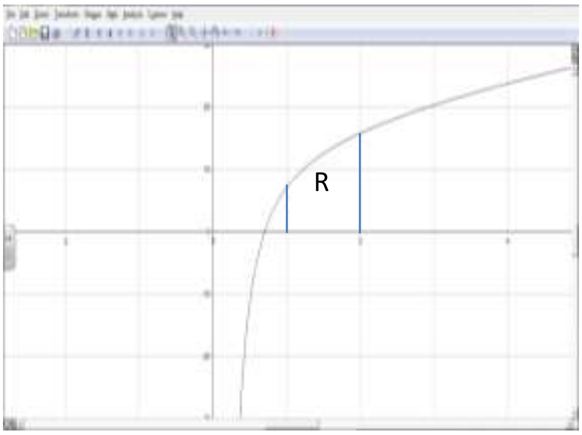
Definite Integration & Area under a curve

- Definite integration = $\int_a^b f(x)\,dx$ = $[g(x)]_a^b$ = g(b) – g(a) = a single value

  where g(x) is the function you get when f(x) is integrated. There is no constant, c, here.

- The area of the region bounded by the curve *y = f (x)*, the ordinates *x = a* and *x= b* and the x axis can be found by definite integration: Area = $\int_a^b f(x)\,dx$ when it exists.

Eg 2 Find the area bounded by the x axis, the curve $y = 12\sqrt{x} - \frac{5}{x^2}$ and the ordinates x = 1 and x = 2.



$\int_1^2 12x^{1/2} - 5x^{-2}dx = \left[\frac{12x^{3/2}}{3/2} - \frac{5x^{-1}}{-1}\right]_1^2$

$= \left[8x^{3/2} + \frac{5}{x}\right]_1^2$

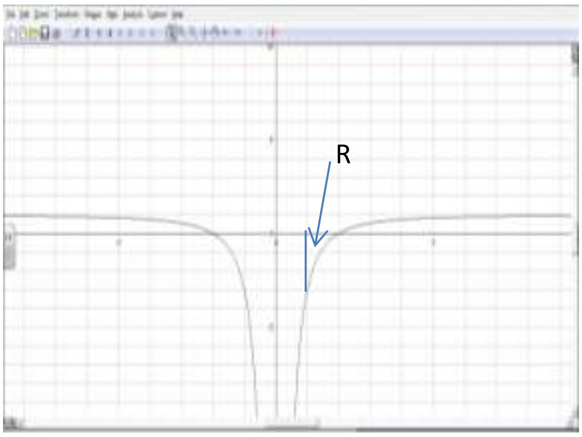
$= \left[8(\sqrt{2})^3 + \frac{5}{2}\right] - \left[8(\sqrt{1})^3 + \frac{5}{1}\right]$

$= [16\sqrt{2} + {}^5/_2] - [13]$

$= 16\sqrt{2} - 10\frac{1}{2}$ [square units]

- Integration will produce a negative answer for areas below the x axis.

Eg 3 Find the area bounded by the x axis, the curve $y = 1 - \frac{4}{x^2}$ and the ordinates x = 1 and x = 2.



$\int_1^2 1 - 4x^{-2}dx = \left[x - \frac{4x^{-1}}{-1}\right]_1^2$

$= \left[x + \frac{4}{x}\right]_1^2$

$= [2 + 2] - [1 + 4]$

$= 4 - 5 = -1$

The minus sign indicates the area is below the x axis
The area is 1 square unit.

- Areas partly above and below the x axis should be integrated separately when integrating with respect to x.

Eg 4 Find the area bounded by the x axis, the curve $y = 1 - \frac{4}{x^2}$ and the ordinates x = 1 and x = 5.

To find the x intercepts solve y = 0: $0 = 1 - \frac{4}{x^2}$ so... $\frac{4}{x^2} = 1$, $x^2 = 4$ and $x = \pm 2$

The graph is below the x axis for $1 \le x < 2$ and above for $2 < x \le 5$

Area above x axis $= \int_2^5 1 - 4x^{-2}dx$

$= \left[x + \frac{4}{x}\right]_2^5$

$= \left[5 + {}^4/_5\right] - [2 + 2] = 1{}^4/_5$

From eg (3) Area below the x axis = 1, hence the area of the region is $2{}^4/_5$

The area between two graphs

- To find the area of the finite region bounded by two curves y = f(x) and y = g(x) you first solve their equations simultaneously to find the limits of integration.

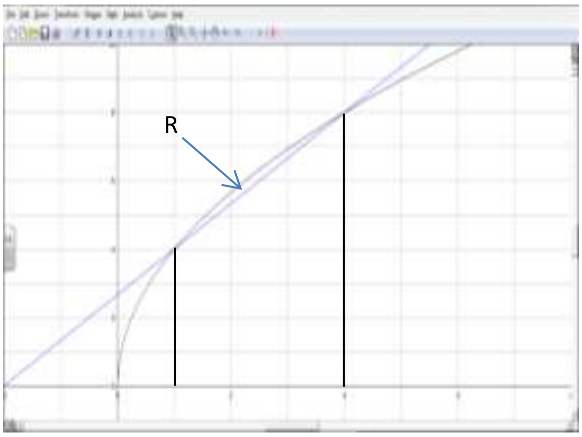
- Assuming y = f(x) is above y = g(x) between the two limits the area would be $A = \int_a^b f(x)dx - \int_a^b g(x)dx$

As the limits of the integrals are the same this can be combined into $A = \int_a^b f(x) - g(x)dx$

- Where one of the functions is a straight line you can find the area bounded by the line, the x axis and the x ordinates using the area of a standard shape such as a triangle or trapezium.

Area of trapezium = ½ (a+b)h (a and b are the lengths of the parallel sides & h is the perpendicular height)

Eg 5 The diagram shows the curve $y = 4x^{1/2}$ and the line $3y = 4x + 8$. Find the area of the finite region bounded by the line and the curve [marked R].



You need to solve the equations $y = 4x^{1/2}$ and $3y = 4x + 8$ simultaneously to find the limits.

$3(4x^{1/2}) = 4x + 8$

$12x^{1/2} = 4x + 8$

$3x^{1/2} = x + 2$

This is a 'disguised quadratic'….Let $p = x^{1/2}$

$3p = p^2 + 2$

$p^2 - 3p + 2 = 0$

$(p - 2)(p - 1) = 0$

$p = 2$ and $p = 1$

As $p = x^{1/2}$, $x = p^2$ Hence $x = 4$ and $x = 1$

Integrating the curve between $x = 1$ and $x = 4$ gives the area under the curve.
Region R is found by finding the area of a trapezium subtracted from the area under the curve.

For the trapezium: Finding a: When $x = 1$ $y = 4x^{1/2} = 4$

Finding b: When $x = 4$ $y = 4x^{1/2} = 8$

Finding h: $h = 4 - 1 = 3$

Area of trapezium = $\frac{1}{2}(4 + 8) \times 3 = 18$

For the area under the curve:

$$A = \int_1^4 4x^{1/2}\,dx = \left[\frac{4x^{3/2}}{3/2}\right]_1^4 = \left[\frac{8(\sqrt{x})^3}{3}\right]_1^4 = \left[\frac{64}{3}\right] - \left[\frac{8}{3}\right] = \frac{56}{3} = 18\,{}^2/_3$$

Hence area of R = $18\,{}^2/_3 - 18 = {}^2/_3$ of a square unit

- Alternative approaches to example 5: Finding the area of the trapezium by integration…

Area of the trapezium = $\int_1^4 \frac{4}{3}x + \frac{8}{3}dx = \left[\frac{2x^2}{3} + \frac{8x}{3}\right]_1^4 = \left[\frac{32}{3} + \frac{32}{3}\right] - \left[\frac{2}{3} + \frac{8}{3}\right] = \frac{64-10}{3} = 18$

Area under the curve = $\int_1^4 4x^{1/2}\,dx = \left[\frac{4x^{3/2}}{3/2}\right]_1^4 = \left[\frac{8(\sqrt{x})^3}{3}\right]_1^4 = \left[\frac{64}{3}\right] - \left[\frac{8}{3}\right] = \frac{56}{3} = 18\,{}^2/_3$

Hence area of R = $18\,{}^2/_3 - 18 = {}^2/_3$ of a square unit

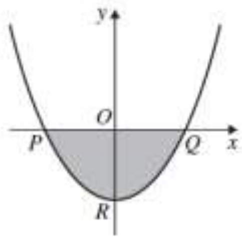
Or by combining the integrations into one [curve – line as curve is above the line] to give…

$$\text{Area of R} = \int_1^4 4x^{1/2} - \frac{4}{3}x - \frac{8}{3}dx = \left[\frac{4x^{3/2}}{3/2} - \frac{2x^2}{3} - \frac{8x}{3}\right]_1^4 = \left[\frac{8(\sqrt{x})^3}{3} - \frac{2x^2}{3} - \frac{8x}{3}\right]_1^4$$

$$= \left[\frac{64}{3} - \frac{32}{3} - \frac{32}{3}\right] - \left[\frac{8}{3} - \frac{2}{3} - \frac{8}{3}\right] = {}^2/_3$$

SKILLS CHECK **4B EXTRA:** Definite integration and areas under curves

1 The diagram shows a sketch of $y = x^2 - 4$.



a Find the coordinates of P, Q and R.

b Find the area of the region enclosed by the curve and the x-axis.

2 a Sketch the curve $y = x(3 - x)$.

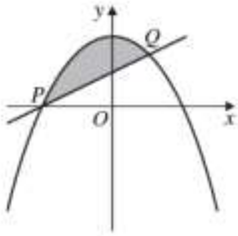
b Find the area of the region enclosed by the curve and the x-axis.

3 a Sketch the line, $x + y = 2$.

b Using integration, find the area of the region R bounded by the coordinate axes and the line $x + y = 2$.

c Check your answer using geometry.

4 The curve $y = 9 - x^2$ and the line $y = 2x + 6$ intersect at P and Q.



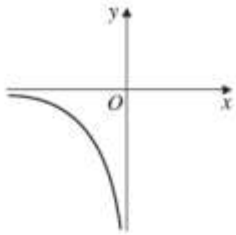
a Find the coordinates of P and Q.

b Find the area of the region enclosed by the line and the curve.

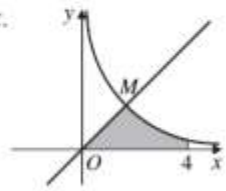
5 Find the area of the region enclosed between the line $y = 3$ and the curve $y = x^2 + 2$.

6 The diagram shows a sketch of $y = -\dfrac{3}{x^2}$, for $x < 0$.

Find the area of the region bounded by the curve, the x-axis and the lines $x = -1$ and $x = -2$.



7 The diagram shows a sketch of the curve $y = \dfrac{1}{\sqrt{x}}$ and the line $y = x$.

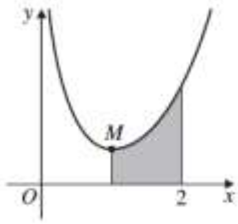


The line and the curve intersect at M.

a Find the coordinates of M.

b Find the area of the shaded region.

8 The diagram shows a sketch of $y = 2x + \dfrac{1}{x^2}$.



There is a minimum turning point at M.

a Find the coordinates of M.

b Find the area of the shaded region.

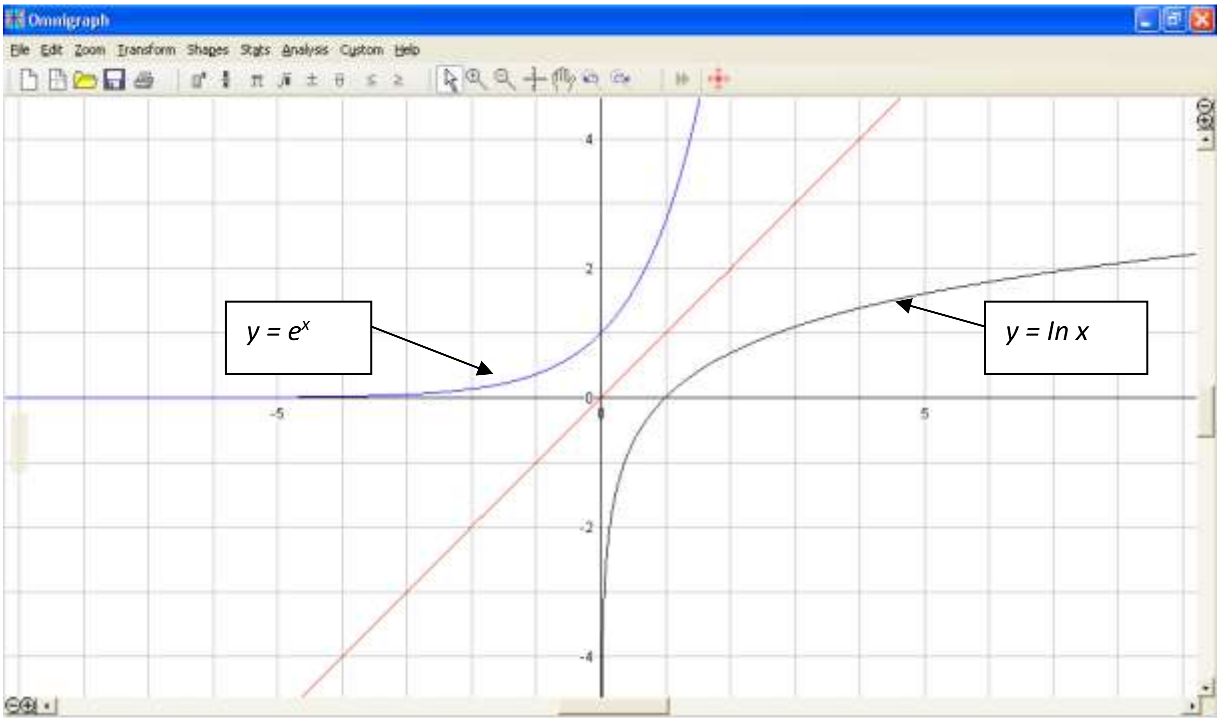
9 Evaluate the following.

a $\displaystyle\int_0^2 2x\,dx$ **b** $\displaystyle\int_{30}^{40} 5\,dx$ **c** $\displaystyle\int_1^2 \tfrac{1}{4}x^3\,dx$

5. Exponentials & Logarithms

The exponential function and the natural logarithm function.

- An exponential function is one of the form $y = a^x$ where 'a' is a constant.
- Exponential graphs all have the same shape and all pass through (0,1). Any exponential function is greater than 0 for all x.



- $y = e^x$ is called 'the' exponential function that has gradient defined to be equal to 1 at the point (0, 1). This property allows $y = e^x$ to be differentiated exactly. e = an irrational number 2.718281828…..
- $y = \ln x = \log_e x$ is the natural logarithm function. It is the inverse of e^x.
- The graph is a reflection of the graph of $y=e^x$ in the line $y=x$

Laws for Logarithms

- $a^c = b$ is identical and interchangeable with $c = \log_a b$
- $\log_a xy = \log_a x + \log_a y$
- $\log_a(x/y) = \log_a x - \log_a y$
- $\log_a x^n = n\log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

Note: Because $\log_a x + \log_a y$ is equal to $\log_a xy$ it means it cannot be equal to $\log_a (x + y)$

Hence $\log_a (x + y)$ does NOT equal $\log_a x + \log_a y$ and you CANNOT 'expand' a log like you would a bracket.

Exponentials and logarithms as inverse functions

- e^x and $\ln x$ are inverse functions in the same way that x^2 and $\sqrt{x}$ are inverses. $\sqrt{(x^2)}$ and $(\sqrt{x})^2$ cancel to give x. In the same way $e^{\ln x}$ and $\ln [e^x]$ cancel to give x.

- ***When you apply a function to an equation you apply it to whole sides not individual terms.***

Eg $\sqrt{y} = x + 4$ — Square each side - <u>not</u> each individual term
$[\sqrt{y}]^2 = [x + 4]^2$

$\ln x = \ln y + 3$ — Apply $e^{\square}$ to each side – <u>not</u> each individual term
$e^{\ln x} = e^{\ln y + 3}$ — Use laws for indices to break up the right hand side
$e^{\ln x} = e^{\ln y} \times e^3$
$x = ye^3$

$\ln x = \ln y + 3$ — Alternatively...Collect the logs on one side
$\ln x - \ln y = 3$ — Use laws for logs to combine into a single log
$\ln [^x/_y] = 3$ — Apply $e^{\square}$ to each side
$e^{\ln [x/y]} = e^3$
$^x/_y = e^3$
$x = ye^3$

- You cannot cancel $e^{\square}$ and ln if there is something in the way – $e^{2\ln y}$ doesn't cancel as 2 is in the way!

EITHER $x = 2 \ln y$ — Pull the 2 inside the ln y then apply $e^{\square}$ to each side
$x = \ln y^2$
$e^x = y^2$
$y = [e^x]^{½} = e^{½x}$

OR $x = 2 \ln y$ — divide by 2 at the start then apply $e^{\square}$ to each side
$^x/_2 = \ln y$
$y = e^{½x}$

- Finally expanding brackets works when you multiply a bracket by a value. Applying a function to a bracket is NOT multiplying the bracket by that function so you cannot 'multiply out' functions like you would expand brackets...

Eg $e^{(a + b)}$ is $e^a \times e^b$ using laws for logs. You ***cannot*** 'expand the bracket' to get $e^a + e^b$.
ln (a + b) is ***NOT*** ln a + ln b in the same way that $\sqrt{(a + b)}$ is ***NOT*** $\sqrt{a} + \sqrt{b}$!

Solving exponential equations

- Basic type

$2^{3-x} = 7^{x+2}$
$\ln [2^{3-x}] = \ln [7^{x+2}]$ — Apply logs to both sides
$(3 - x) \ln 2 = (x + 2) \ln 7$ — Use laws for logs to bring out the power
$3 \ln 2 - x \ln 2 = x \ln 7 + 2 \ln 7$ — Expand and isolate the terms in x
$3 \ln 2 - 2 \ln 7 = x \ln 7 + x \ln 2$ — Take x out as a common factor
$3 \ln 2 - 2 \ln 7 = x (\ln 7 + \ln 2)$ — Divide through to find x.
$\dfrac{3 \ln 2 - 2 \ln 7}{(\ln 7 + \ln 2)} = x$

- Quadratic type

$2^{2x+3} - 10(2^x) + 3 = 0$ — Express 2^{2x+3} in terms of 2^x
$(2^x)^2 \times 2^3 - 10(2^x) + 3 = 0$ — Let $y = 2^x$ and write as a quadratic
$8y^2 - 10y + 3 = 0$ — Factorise & solve for y
$(4y - 3)(2y - 1) = 0$
$y = ¾ , \quad y = ½$ — Remember $y = 2^x$ so solve for x using logs or inspection if possible
$2^x = ¾ , \quad 2^x = ½$
$\ln 2^x = \ln [¾] \quad x = -1$
$x \ln 2 = \ln [¾]$
$x = \ln [¾] / \ln 2$

- Disguised quadratic $8(2^x) + 3(2^{-x}) = 10$ — $2^{-x} = {}^1/2^x$ then multiply through to give the exact same quadratic as the example above.

SKILLS CHECK 1C EXTRA: Exponentials, logarithms and exponential growth and decay

1 Find the exact solution of $e^{2x+3} = 5$.

2 Solve $e^{4x} - e^{2x} = 0$.

3 Find the exact value of y for which $\ln(2y - 5) - \ln y + 3 = 0$.

4 Solve $\ln(2 - 3x) = -1$, giving your answer to 3 s.f.

5 Solve the equation $\ln\sqrt{x - 1} = 5$, giving your answer in the form $e^a + b$, where a and b are integers to be found.

6. Functions

Introduction to functions and mappings.

- A function maps an element of the domain (x) to an element of the range (y) so y = f(x).

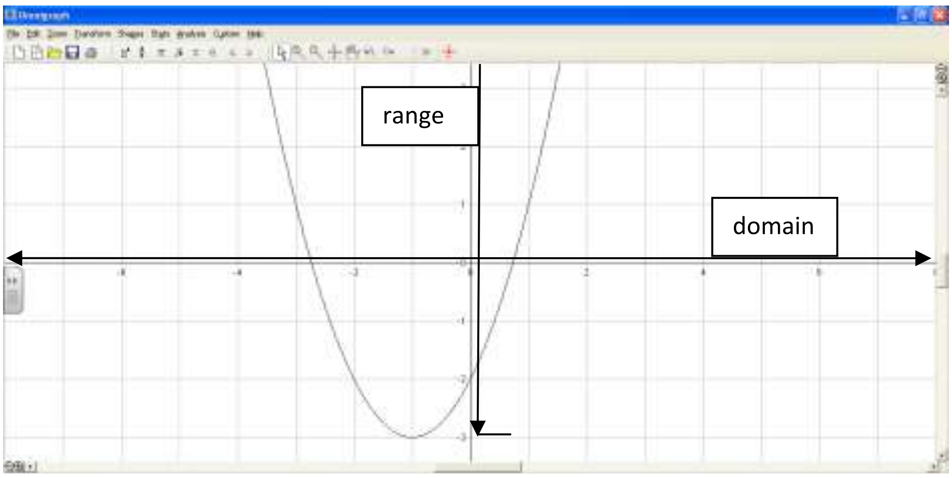
- A function can be '1 to 1' ....such as f(x) = 2x + 3 which maps each x value to one y value
  It can also be 'many to 1' ...such as $y = x^2$ where different x values can map to the same y value.

- A function cannot be 'one to many'...where an x value maps to more than 1 y value.

Domain and range

- The domain is the set of x values that the function maps to y values. The domain is expressed in terms of x.
- The range is the set of y values that the function maps to. The range is expressed in terms of y or f(x).

  Eg $f(x) = (x + 1)^2 - 3$



  The domain of the function is x ϵ IR [x is any element of the real numbers]
  The range is f(x) ϵ IR , f(x) ≥ -3

- When the domain for a function is restricted the range does not necessarily come from using just the end points of its domain.

  Eg: $f(x) = (x + 1)^2 – 3$ is defined for -2 < x < 1

  Using the end points... f(-2) = -2 and f(1) = 1. This might make you think the range is -2 < f(x) < 1 but this is wrong!

  If you look at the graph (above) and take into account the minimum point you can see that the range is actually -3 ≤ f(x) < 1.

  When considering the range, draw a diagram and consider stationary points – for a quadratic you often complete the square to find the min / max point.

- When writing a function you must state the function f(x) = … and the domain x … .
  Egs $y = e^x$ x ϵ IR
  y = ln x x > 0, x ϵ IR [ln is not defined for x = 0 or x < 0]
  $y = {}^3/x$ x ≠ 0, x ϵ IR [You can't divide by 0 hence x ≠ 0]
  y = tan x $x \neq (2n+1)^{\pi}/_2$, x ϵ IR [tan x = sinx / cosx is not defined wherever cosx = 0 ie the odd numbered multiples of π/2]

Composite functions

- Most functions are a combination of more basic functions. These are called composite functions.
- gh(x) means g[h(x)] - ie apply h first then g
- Generally gh(x) is not the same as hg(x)

Eg $g(x) = x^2 - 4$ $x \in IR$ and $h(x) = {}^{5}/_{x}$ $x \neq 0$, $x \in IR$

$gh(x) = g[h(x)]$
$= g[{}^{5}/_{x}]$
$= [{}^{5}/_{x}]^2 - 4$
$= {}^{25}/_{x^2} - 4$

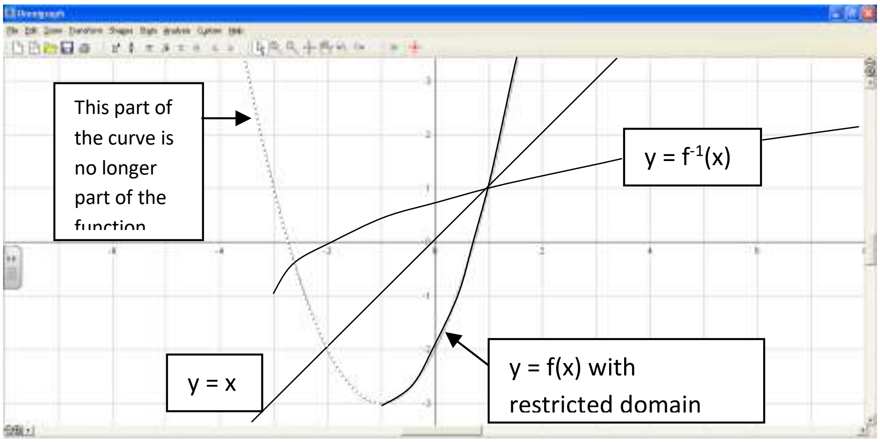
$hg(x) = h[g(x)]$
$= h[x^2 - 4]$
$= {}^{5}/_{[x^2 - 4]}$

- Consider the domain of the composite functions: the domain of gh(x) is the same as the domain of h. However the domain of hg(x) is not the same as the domain of g.
  The domain of hg(x) must be $x \neq \pm 2$, $x \in IR$. In this case the domain of g has to be restricted to avoid g mapping to 0 which is not in the domain of h. [Note this means -5/4 will not be in the range of hg(x)].

- You could be asked to find a composite function [eg find hg(x)], the result of a composite function for a specific value [eg find hg(3)] or to solve an equation involving composite functions [eg solve hg(x) = 3].

Inverse functions

- Inverse functions need to be one to one mappings.
- A many to one function can only have an inverse if its domain is restricted to make it one to one.
  Eg $y = (x + 1)^2 - 3$ $x \in IR$ is many to one so does not have an inverse
  However if you restrict its domain to $y = (x + 1)^2 - 3$ $x > -1$, $x \in IR$ then it becomes one to one [as it's minimum point is (-1, -3)]



- The relationship between a function and its inverse can be shown graphically – each is a reflection of the other in the line $y = x$ $\in IR$
- The range of f(x) is the domain of $f^{-1}(x)$ and the range of $f^{-1}(x)$ is the domain of f(x)

- The method for finding the inverse function: $y = f(x)$ so $x = f^{-1}(y)$, hence make x the subject of the formula...

$y = (x + 1)^2 - 3$
$y + 3 = (x + 1)^2$
$\pm\sqrt{(y + 3)} = x + 1$
$-1 + \sqrt{(y + 3)} = x$
$-1 + \sqrt{(y + 3)} = f^{-1}(y)$

Note:
(1) you need to determine if + or – square root using the domain of f(x).
As the domain of f(x) is $x > -1$ you need the positive square root]
(2) The domain of the inverse function must be stated – it is the same as the range of f(x)

Using a change of variable.... $f^{-1}(x) = -1 + \sqrt{(x + 3)}$, $x > -3$ $x \in IR$

- Some functions such as $f(x) = {}^{1}/_{x}$ are self inverses.

SKILLS CHECK 1A EXTRA: Functions

1 State whether each of the following mappings represents a function. If it is a function, state whether it is one-one or many-one.

a $y = \frac{7 + 2x}{4}, x \in \mathbb{R}$

b $y = x^3, x \in \mathbb{R}$

c $x^2 - y^2 = 4, x \in \mathbb{R}$

d $y = \pm\sqrt{x}, x \in \mathbb{R}, x \geqslant 0$

2 Find the range of each of the following functions:

a $f(x) = x^2 + 4, x = \{-2, -1, 0, 1, 2\}$

b $g: x \mapsto \frac{1}{2x + 1}, x = \{-1, 0, 1, 2, 3\}$

c $f: x \mapsto -x^3, x \in \mathbb{R}$

d $f(x) = (x - 2)^2 + 7, x \in \mathbb{R}$

3 For each of the following functions

i sketch the function,

ii state its range.

a $f(x) = (x + 3)^2, x \in \mathbb{R}$

b $g: x \mapsto 2^x, x \in \mathbb{R}$

c $f: x \mapsto \cos x, x \in \mathbb{R}, 0 \leqslant x \leqslant \frac{\pi}{2}$

d $h(x) = \begin{cases} -x, & x \in \mathbb{R}, x \leqslant 1 \\ x - 2, & x \in \mathbb{R}, x > 1 \end{cases}$

4 The function f is defined by $f: x \mapsto (2x + 1)^2$ for the domain $-3 \leqslant x \leqslant 1$.
Find the range of f.

---

5 Functions f, g and h are defined as follows:

$f: x \mapsto \log x, x \in \mathbb{R}, x > 0$ $\quad$ $g: x \mapsto x^3, x \in \mathbb{R}$ $\quad$ $h: x \mapsto 2x + 1, x \in \mathbb{R}$

Find the value of

a gh(2) **b** fg(1) **c** gg(−2)

d fh(3) **e** hhg($\frac{1}{2}$) **f** hf(5)

6 Functions f, g and h are defined as follows:

$f: x \mapsto 2^{x+1}, x \in \mathbb{R}$ $\quad$ $g: x \mapsto \frac{x + 2}{3}, x \in \mathbb{R}$ $\quad$ $h: x \mapsto 1 - x, x \in \mathbb{R}$

Find the composite functions

a gh **b** hf **c** fh **d** gg

7 Functions f, g and h are defined as follows:

$f(x) = 3x - 1, x \in \mathbb{R}$ $\quad$ $g(x) = x^2 + 4, x \in \mathbb{R}$ $\quad$ $h(x) = 3^x, x \in \mathbb{R}$

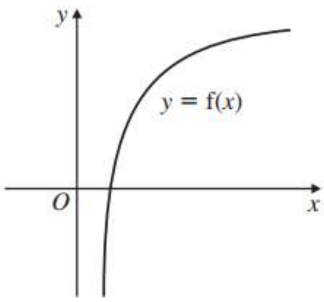
Solve these equations:

a $ff(x) = 8$ **b** $fh(x) = 8$ **c** $fg(x) = 19\frac{1}{3}$ **d** $gf(x) = 9$

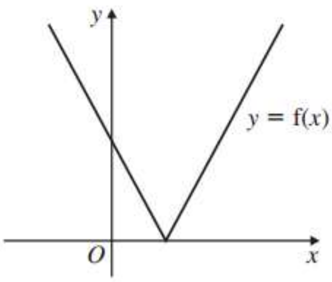
---

8 Which of the following functions has an inverse?

a



b



9 For each of the following functions, f,

i find the inverse function f^{-1}, stating its domain and range,

ii on the same set of axes, sketch $y = f(x)$ and $y = f^{-1}(x)$.

a $f: x \mapsto 5(2x + 1), x \in \mathbb{R}$

b $f: x \mapsto x^3, x \in \mathbb{R}$

c $f: x \mapsto \log(x - 1), x \in \mathbb{R}, x > 1$

d $f: x \mapsto \frac{1}{x}, x \in \mathbb{R}, x \neq 0$

10 Functions f, g and h are defined as follows:

$f(x) = 8 - x, x \in \mathbb{R}$ $\quad g(x) = \frac{1}{x}, x \in \mathbb{R}, x \neq 0$ $\quad h(x) = 5x - 1, x \in \mathbb{R}$

Solve these equations:

a $ff(x) = h(x)$

b $f(x) = h^{-1}(x)$

c $gf(x) = f(x)$

11 A function is defined by $f: x \mapsto (x + 1)^2 - 1, x \in \mathbb{R}$.

a State the range of f.

b Find $f^{-1}(x)$ and state its domain and range.

c Find the values of x for which $f(x) = f^{-1}(x)$.

---

7 Rational Functions (Algebraic Fractions)

Simplifying fractions by cancelling common factors

- Factorise everything first!
- 'You can only cancel when everything is multiplied together' - you usually have to factorise to ensure this is the case. Look out for difference of two squares.

$$\frac{x^2 + 4x + 3}{2x^2 - 18} = \frac{(x+1)\cancel{(x+3)}}{2\cancel{(x+3)}(x-3)} = \frac{(x+1)}{2(x-3)}$$

Multiplication & Division of fractions

- Factorise everything first! For multiplication you multiply the numerators together and multiply the denominators together and cancel in your working.

- For division you must multiply the first fraction by the reciprocal of the second fraction. (ie.turn the second fraction upside down and then multiply by the first fraction.)

$$\frac{x^2 + x}{3x - 6} \div \frac{x^2 - 1}{6x - 12}$$
$$= \frac{x(x+1)}{3(x-2)} \div \frac{(x+1)(x-1)}{6(x-2)}$$
$$= \frac{x(x+1)}{3(x-2)} \times \frac{6(x-2)}{(x+1)(x-1)} = \frac{2x}{x-1}$$

Addition & Subtraction of fractions

- Factorise everything first! Put the fractions over the lowest common denominator using equivalent fractions and add / subtract the numerators

- Be careful to use brackets for questions involving subtraction – so that you subtract <u>all</u> the terms you should subtract and not just the first of those terms.

- DO NOT USE THE METHOD OF CROSS MULTIPLICATION AS IT CAN MAKE THESE QUESTIONS <u>VERY</u> DIFFICULT.

$$\frac{x-2}{x-4}-\frac{3x+4}{x^2-16}$$

$$=\frac{x-2}{x-4}-\frac{3x+4}{(x-4)(x+4)}$$

Factorise and then see that the lowest common denominator will be (x – 4)(x + 4)

$$=\frac{(x-2)(x+4)}{(x-4)(x+4)}-\frac{3x+4}{(x-4)(x+4)}$$

Use equivalent fractions to write over the common denominator is (x – 4)(x + 4)

$$=\frac{(x^2+2x-8)-(3x+4)}{(x-4)(x+4)}$$

Subtract both the 3x and the 4. Forgetting this is the most common error in fractions!

$$=\frac{x^2+2x-8-3x-4}{(x-4)(x+4)}$$

$$=\frac{x^2-x-12}{(x-4)(x+4)}$$

Don't expand the denominator. Instead try to factorise the numerator and see if you can simplify the fraction.

$$=\frac{(x-4)(x+3)}{(x-4)(x+4)}=\frac{x+3}{x+4}$$

8. <u>Polynomial Division</u>

Polynomials & Algebraic Division

- A polynomial is an expression of the form: $a_n x^n + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ where the a_i are constants
- The degree of a polynomial is the highest power of the variable in the polynomial
  - An improper fraction is a fraction whose denominator has degree equal to or greater than the degree of its denominator. Algebraic division can be used to turn an improper fraction into a mixed polynomial.

$\frac{x^3-x^2-4x-5}{x-2}$ is an improper fraction

This example shows how to divide $x^3 - x^2 - 4x - 5$ by x - 2…

The quotient is $x^2 + x - 2$
The remainder is –9

$\frac{x^3-x^2-4x-5}{x-2}$ is the same as $x^2 + x - 2 - \frac{9}{x-2}$

You can also write: $x^3 - x^2 - 4x - 5 = (x - 2)(x^2 + x - 2) - 9$

$$\begin{array}{r}
x^2 + x - 2 \\
x-2\,\big)\,\overline{x^3 - x^2 - 4x - 5} \\
\underline{(x^3 - 2x^2)} \qquad\qquad \\
x^2 - 4x \qquad \\
\underline{-(x^2 - 2x)} \qquad \\
-2x - 5 \\
\underline{-(-2x + 4)} \\
-9
\end{array}$$

Comparing coefficients

- An alternative to algebraic division is the method of comparing coefficients: If you are dividing a cubic by a linear expression you will get a quadratic expression with remainder.

$x^3 - x^2 - 4x - 5 \equiv (x-2)(ax^2 + bx + c) + r$

Compare coefficient of	x^3:	$1 \equiv a$				
	x^2:	$-1 \equiv b - 2a$	hence	$-1 \equiv b - 2$	so	$b \equiv 1$
	x:	$-4 \equiv c - 2b$	hence	$-4 \equiv c - 2$	so	$c \equiv -2$
	x^0:	$-5 \equiv -2c + r$	hence	$-5 \equiv 4 + r$	so	$r \equiv -9$

Hence $x^3 - x^2 - 4x - 5 = (x-2)(x^2 + x - 2) - 9$

Factor Theorems

- If f(x) is a polynomial, if for $x = a$ we have $f(a) = 0$ then $(x - a)$ is a factor of f(x).
- If f(x) is a polynomial, if for $x = {}^a/_b$ we have $f({}^a/_b) = 0$ then $(bx - a)$ is a factor of f(x).

Question Types

- Express an improper fraction as a mixed polynomial expression

- Show $(2x - 1)$ is a factor of P(x)

- Solve a cubic equation and sketch a cubic graph. Hence solve a cubic inequality

- If P(x) is a cubic with two unknown coefficients, a and b, use information about factors and remainders to form and solve two simultaneous equations to find a and b

Examples

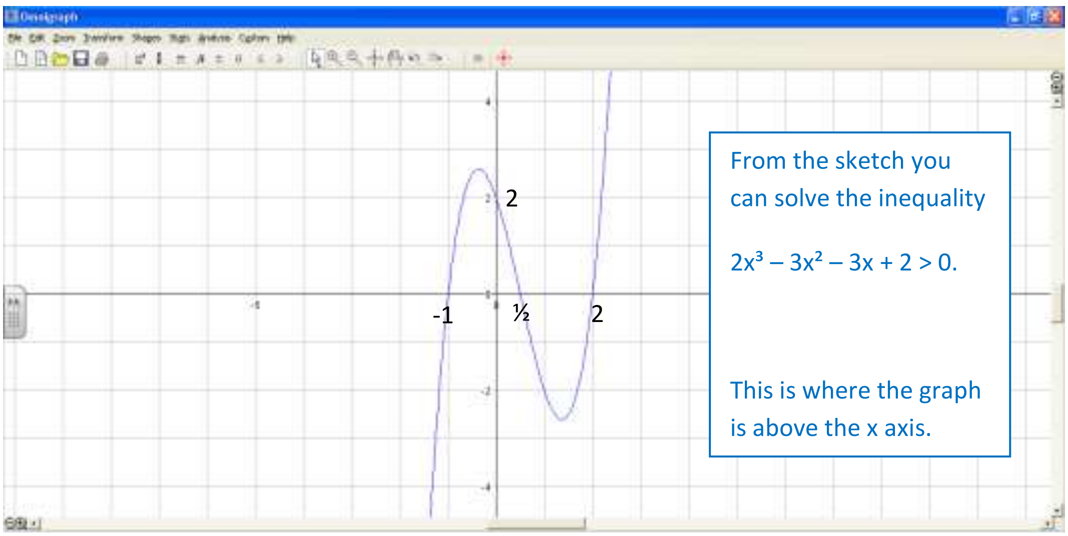
[1] Show $(2x - 1)$ is a factor of $f(x) = 2x^3 - 3x^2 - 3x + 2$. Hence write f(x) as a product of three linear factors

$f(½) = 2(½)^3 - 3(½)^2 - 3(½) + 2 \quad = {}^1/_4 - {}^3/_4 - {}^3/_2 + 2 \quad = 0$
Hence by the factor theorem as $f(-1) = 0$ then $(x + 1)$ is a factor of f(x)

Using algebraic division or comparing coefficients you can show that the quadratic factor is $x^2 - x - 2$
Hence $f(x) = (2x-1)(x^2 - x - 2)$
$f(x) = (2x-1)(x-2)(x+1)$ as a product of 3 linear factors

[2] Sketch $y = 2x^3 - 3x^2 - 3x + 2$. Hence solve $2x^3 - 3x^2 - 3x + 2 > 0$.

When $x = 0$... $y = 2$
This is the same cubic as in [2] hence $y = (x+1)(2x-1)(x-2)$
When $y = 0$... $x = -1$, $x = ½$ and $x = 2$



[3] Show that the only solution to $x^3 - 2x^2 + 4x - 3 = 0$ is $x = 1$.

If $x = 1$ is a solution then $(x - 1)$ is a factor. Using algebraic division or comparing coefficients you can show that $x^3 - 2x^2 - 2x + 3 = 0$ is the same as $(x - 1)(x^2 - x + 3) = 0$

$x = 1$ from the first bracket is clearly a solution.

Use the discriminant: $b^2 - 4ac = (-1)^2 - 4(1)(3) = -11 < 0$ hence $x^2 - x - 3 = 0$ has no solutions.

INSERT SKILLS EXTRA 1A core 4

SKILLS CHECK 1A EXTRA: Simplifying rational expressions

1 Simplify fully:

a $\dfrac{x^2 - 1}{2x - 2}$ **b** $\dfrac{x^2 + 8x + 15}{x^2 + 4x - 5}$ **c** $\dfrac{4x^2 - 8x + 3}{2x^2 + 13x - 7}$ **d** $\dfrac{x^3 - 25x}{2x^3 + 10x^2}$

2 Express as a single fraction in its simplest form:

a $\dfrac{x^2 - 5x}{x^3} \times \dfrac{2x^2}{4x - 20}$ **b** $\dfrac{3x^2}{6x + 12} \div \dfrac{x^2 - 2x}{x^2 - 4}$

c $\dfrac{2y^2 + 3y}{y^2 + 3y - 4} \div \dfrac{4y^2 - 9}{2y^2 + 5y - 12}$ **d** $\dfrac{3x^2 - 3}{6x^3 + 6x^2} \times \dfrac{2x^3}{4x^2 - x - 3}$

3 Express as a fraction in its simplest form:

a $\dfrac{3}{x - 5} - \dfrac{4}{2x - 1}$ **b** $2 - \dfrac{3}{2t + 1}$

c $\dfrac{5}{x + 3} - \dfrac{2x - 29}{x^2 - x - 12}$ **d** $\dfrac{3}{x^2 - 4} + \dfrac{x}{x^2 + 2x - 8}$

4 Divide $4x^3 - 3x + 9$ by $2x - 1$.

5 a Simplify $\dfrac{2}{x^2 - 1} - \dfrac{7}{2x^2 - 3x - 5}$.

b Hence solve $\dfrac{2}{x^2 - 1} - \dfrac{7}{2x^2 - 3x - 5} = -\dfrac{1}{3}$.

6 Find the quotient and the remainder when $3x^4 - 2x^3 + x - 1$ is divided by $x^2 + 2$.

SKILLS CHECK 3A EXTRA: Factor and remainder theorems

1 For the following polynomials, solve $f(x) = 0$.

a $f(x) = (3x + 2)(x - 1)(4x + 5)$ **b** $f(x) = (x^2 - 16)(x^2 - 25)$

c $f(x) = (3x - 1)(2x + 1)(3x - 2)$ **d** $f(x) = (4x - 5)(x + 3)^2$

2 It is given that $f(x) = x^3 - 4x^2 + x + 6$.

a Show that $(x + 1)$ is a factor of $f(x)$.

b Write $f(x)$ as the product of three linear factors.

c Solve $f(x) = 0$.

3 Factorise $x^3 + 2x^2 - 4x - 8$, given that $x = 2$ is a root of $x^3 + 2x^2 - 4x - 8 = 0$.

6 Find the roots of the equation $x^3 - x^2 - 10x + 12 = 0$, given that 3 is a root. Give your answers in surd form if appropriate.

7 Use the factor theorem to find a linear factor of $f(x)$ where $f(x) = x^3 + 5x^2 + 3x - 9$. Hence express $f(x)$ as a product of three linear factors.

10 It is given that $f(x) = x^3 - 7x + 6$.

a Show that $(x - 2)$ is a factor of $f(x)$.

b Show that $(x + 3)$ is also a factor of $f(x)$.

c Factorise $f(x)$ completely.

The graph of $y = f(x)$ crosses the y-axis at A.

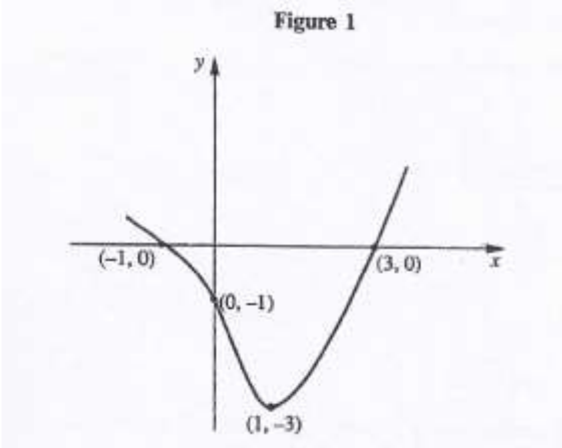
d Find the coordinates of A.

e Find the coordinates of the points where the curve crosses the x-axis.

f Sketch $y = f(x)$.

Section B: Assessment

Question 1: Transformations



The graph above shows a sketch of the curve with equation y = f(x).
The curve crosses the coordinate axes at the points (-1,0), (0,-1) and (3,0).
The minimum point is (1, -3).
On separate diagrams sketch the curve with equation:

a) y = f(x + 1) [3]
b) y = f($\frac{1}{2}x$) [3]

On each diagram, show clearly the coordinates of the minimum point, and each of point at which the curve crosses the coordinate axes.

Question 2: Trigonometry

a) Show that the equation

$$3\sin^2 x = 5 \cos x + 1$$

can be written as

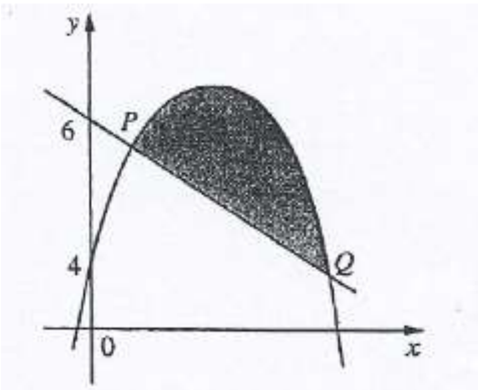
$$3\cos^2 x + 5\cos x - 2 = 0$$ [2]

b) Hence solve, for $0 \leq x < 360^o$, the equation

$$3\sin^2 x = 5 \cos x + 1 ,$$

giving your answers to 1 decimal place where appropriate. [5]

Question 3: Integration



The line with equation y = 6 – x cuts the curve y = 4 + 2x – x^2 at the points P and Q, as shown.

a) Find the coordinates of P and Q. [5]
b) Find the area of the shaded region between the line and the curve as shown in the diagram. [7]

Question 4: Integration and Differentiation

The curve C with equation y = f(x) is such that

$$\frac{dy}{dx} = 4x + \frac{4}{\sqrt{x}}, \quad x > 0$$

a) Show that, when x = 2, the exact value of $\frac{dy}{dx}$ is $8 + 2\sqrt{2}$. [3]

The curve C passes through the point (4,50).

b) Find f(x). [6]

Question 5: Differentiation

The curve C has the equation

$$y = 6 - \frac{1}{x^2} - 2x\sqrt{x}, \quad x > 0.$$

a) Write $x\sqrt{x}$ in the form x^n, where n is a fraction. [1]

b) Find $\frac{dy}{dx}$. [3]

c) Find an equation of the normal to the curve C at the point on the curve where x = 1. [4]

d) Find $\frac{d^2y}{dx^2}$. [2]

e) Deduce that the curve C has no minimum points. [2]

Question 6: Exponentials and Logarithms

Solve each equation, giving your answers in exact form.

a) $\ln(4x + 1) = 2$ [3]

b) $3e^x + 2e^{-x} = 7$ [5]

Question 7: Exponentials

a) Sketch the graph $y = e^{ax + b}$, given a and b > 0.
Mark the coordinates of the point where the graph meets either the x-axis or the y-axis. [2]

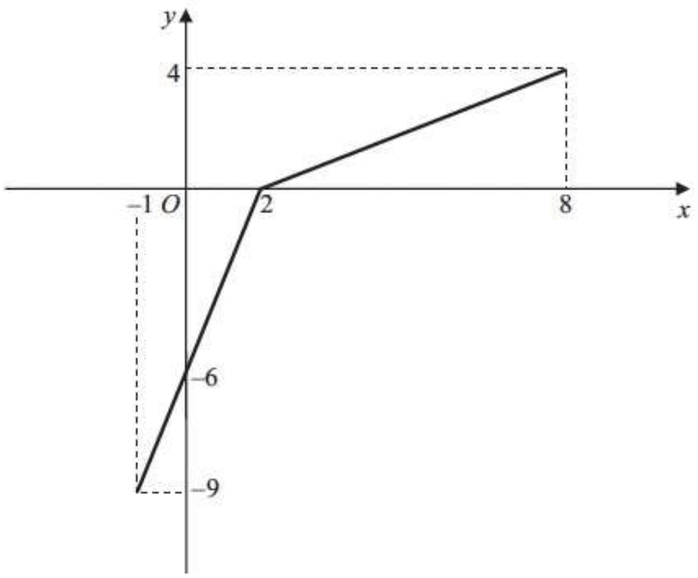
b) Given that when x = 0, y = 4, find the exact value of b. [2]

Question 8: Functions

The function f is defined by

$$f: x \to \frac{3 - 2x}{x - 5} \quad x \in \mathbb{R}, x \neq 5$$

a) Find $f^{-1}(x)$ [3]



The function g has domain $-1 \leq x \leq 8$ and is linear from (-1,-9) to (2,0) and from (2,0) to (8,4), The above diagram shows a sketch of the graph of y=g(x)

b) Write down the range of g. [1]
c) Find gg(2) [2]
d) Find fg(8) [2]
e) State the domain of the inverse function g^{-1}. [1]

Question 9: Factor Theorem

$$f(x) = 2x^3 - 7x^2 - 10x + 24$$

a) Use the factor theorem to show that (x + 2) is a factor of f(x). [2]
b) Factorise f(x) completely. [4]

Question 10: Rational Functions

Express

$$\frac{2(3x+2)}{9x^2-4} - \frac{2}{3x+1}$$

as a single fraction in its simplest form. [4]

Total marks = 77

Section A: Answers

1. Transformations

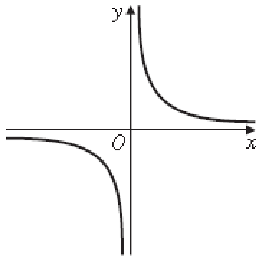
SKILLS CHECK 3G EXTRA: Answers

1 **a** Translation by $\begin{pmatrix}2\\0\end{pmatrix}$ **b** Stretch scale factor 3 in y-direction

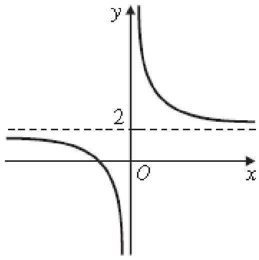
2 **a** $A(-1, 0)$ $B(0, 2)$ $C(1, 0)$ **b** $A(-1, 0)$ $B(0, -1)$ $C(1, 0)$ **c** $A(-\frac{1}{2}, 0)$ $B(0, 1)$ $C(\frac{1}{2}, 0)$

3 **a** Translate by $\begin{pmatrix}2\\3\end{pmatrix}$ **b** Translate by $\begin{pmatrix}-4\\-10\end{pmatrix}$

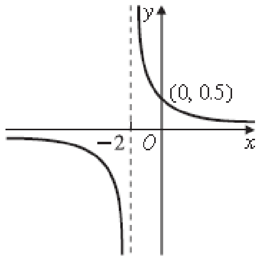
4 **a**



b



c



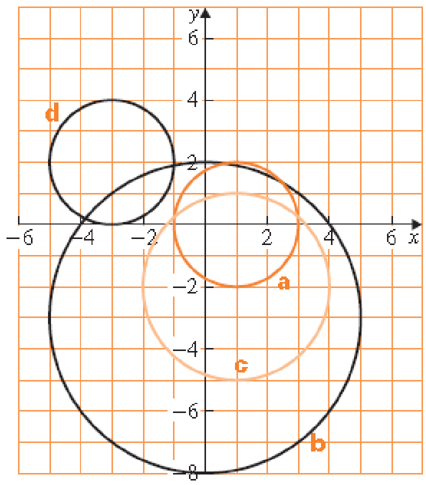
5 Translate by $\begin{pmatrix}4\\0\end{pmatrix}$

6 **a**

7 **a** $y = \frac{1}{2}f(x)$ **b** $y = f(x - 1)$ **c** $y = -f(x)$ **d** $y = f(2x)$

8 **a** $(x - 1)^2 + y^2 = 4$ **b** $x^2 + (y + 3)^2 = 25$

c $(x - 1)^2 + (y + 2)^2 = 9$ **d** $(x + 3)^2 + (y - 2)^2 = 4$

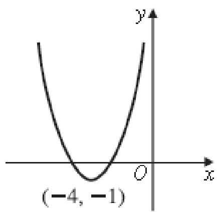


9 **a** **a** $\begin{pmatrix}1\\0\end{pmatrix}$ **b** $\begin{pmatrix}0\\-3\end{pmatrix}$ **c** $\begin{pmatrix}1\\-2\end{pmatrix}$ **d** $\begin{pmatrix}-3\\2\end{pmatrix}$

b **a** $(1, 0), 2$ **b** $(0, -3), 5$ **c** $(1, -2), 3$ **d** $(-3, 2), 2$

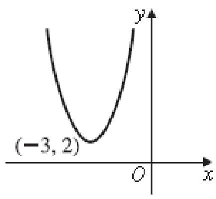
10 **a** $x^2 + (y - 3)^2 = 4$ **b** $(x + 2)^2 + (y + 2)^2 = 2.25$

11 **a**



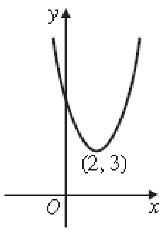
Translation $\begin{pmatrix}-4\\-1\end{pmatrix}$

b



Translation $\begin{pmatrix}-3\\2\end{pmatrix}$

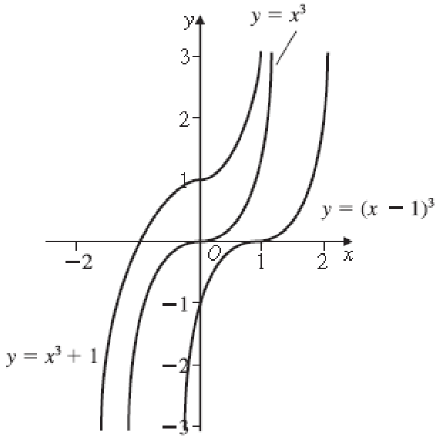
c



Translation $\begin{pmatrix}2\\3\end{pmatrix}$

12 **a** **i** Translation $\begin{pmatrix}1\\0\end{pmatrix}$ **ii** Translation $\begin{pmatrix}0\\1\end{pmatrix}$

b



13 **a** $\frac{1}{4}$ **b** 16

14 **a** Reflection in the x-axis. $y = -\cos x$ **b** Translation $\begin{pmatrix}0\\-1\end{pmatrix}$. $y = \cos x - 1$

c Stretch scale factor 2 in the y-direction. $y = 2\cos x$.

2. Trigonometry

SKILLS CHECK 1C EXTRA: Answers

1 **a** 66°, 294° (nearest degree) **b** 30°, 150°, 210°, 330° **c** 26°, 206° (nearest degree)

2 **a** 75°, 105° **b** 10°, 70°, 130° **c** 60°

3 188.63°, 351.37°

4 0°, 60°, 120°, 180°, 360°

5
$$\tan\theta + \frac{1}{\tan\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$

6
$$\tan^2 x(1+\sin x) = \frac{\sin^2 x(1+\sin x)}{\cos^2 x} = \frac{\sin^2 x(1+\sin x)}{1-\sin^2 x} = \frac{\sin^2 x(1+\sin x)}{(1+\sin x)(1-\sin x)} = \frac{\sin^2 x}{1-\sin x}$$

7 $a = \frac{3}{4}$, $b = -\frac{1}{2}$

8 $-\frac{5}{13}$

9 0, 120°, 360°

10 **a** $3\sin^2 x + \cos^2 x - \cos x = 2$
$3(1 - \cos^2 x) + \cos^2 x - \cos x = 2$
$3 - 3\cos^2 x + \cos^2 x - \cos x = 2$
$2\cos^2 x - \cos x - 1 = 0$

b $60°, 180°, 300°$

3. Differentiation

SKILLS CHECK 4B EXTRA: Answers

1 $y = -12x - 3$

2 $x + 8y - 25 = 0$

3 **a** $(1, 0)$ **b** $\frac{1}{3}$ **c** $3y = x - 1$ **d** $(0, -\frac{1}{3})$ **e** $\frac{1}{6}$ units^2

4 $y + 3x = 3$; both have gradient -3

5 $x < 4.5$

6 $x > \frac{1}{3}$

7 $49x - 16y - 76 = 0$

8 **a** -6 **b** $x - 6y - 31 = 0$

SKILLS CHECK 4C EXTRA: Answers

1 $(3, 9)$, maximum

2 $-2, -\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}, 2$

3 $a = 2, b = 3$

4 15

5 $y = -7$

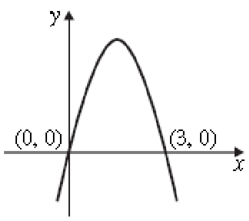
6 $(-3, -55)$ min $(3, 53)$ max

4. Integration

SKILLS CHECK 4B EXTRA: Answers

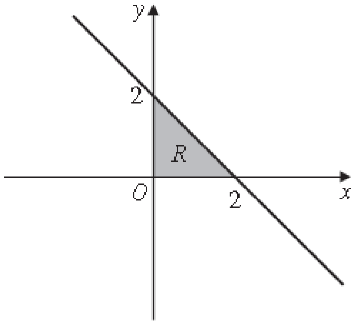
1 **a** $P(-2, 0), Q(2, 0), R(0, -4)$ **b** $10\frac{2}{3}$

2 **a**



b $4\frac{1}{2}$

3 **a**



b 2

4 **a** $P(-3, 0)$, $Q(1, 8)$ **b** $10\frac{2}{3}$

5 $1\frac{1}{3}$

6 1.5

7 **a** $(1, 1)$ **b** 2.5

8 **a** $(1, 3)$ **b** 3.5

9 **a** 4 **b** 50 **c** $\frac{15}{16}$

5. Exponentials and Logarithm

SKILLS CHECK 1C EXTRA: Answers

1 $x = \dfrac{\ln 5 - 3}{2}$

2 $x = 0$

3 $y = \dfrac{5e^3}{2e^3 - 1}$

4 $x = 0.544$

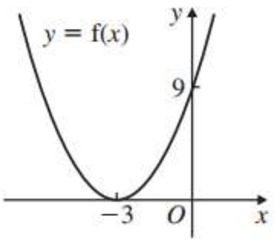
5 $x = e^{10} + 1$

6. Functions

1 **a** one-one function
b one-one function
c not a function
d not a function

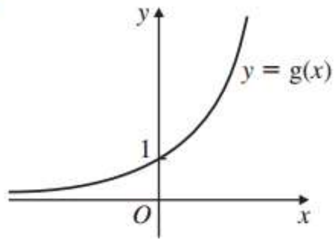
2 **a** $f(x) = \{4, 5, 8\}$
b $g(x) = \{-1, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\}$
c $f(x) \in \mathbb{R}$
d $f(x) \in \mathbb{R}, f(x) \geqslant 7$

3 **a**



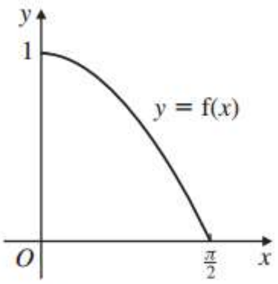
$f(x) \in \mathbb{R}, f(x) \geqslant 0$

b



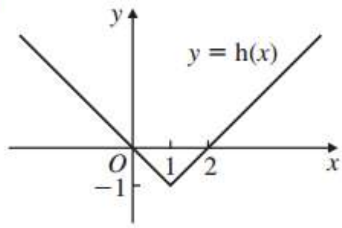
$g(x) \in \mathbb{R}, g(x) \geqslant 0$

c



$f(x) \in \mathbb{R}, 0 \leqslant f(x) \leqslant 1$

d



$h(x) \in \mathbb{R}, h(x) \geqslant -1$

4 $f(x) \in \mathbb{R}, 0 \leqslant f(x) \leqslant 25$

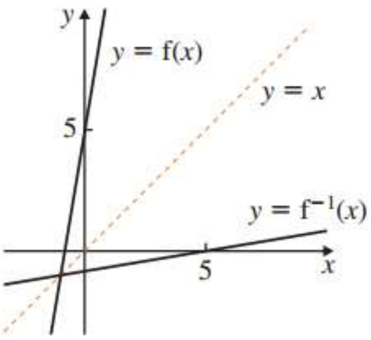
5 **a** 125 **b** 0 **c** -512 **d** 0.85 **e** 3.5 **f** 2.40

6 **a** $\dfrac{3-x}{3}$ **b** $1 - 2^{x+1}$ **c** 2^{2-x} **d** $\dfrac{x+8}{9}$

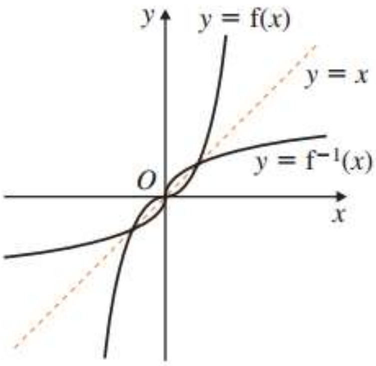
7 **a** $\frac{4}{3}$ **b** 1 **c** $\pm\frac{5}{3}$ **d** -0.412 or 1.08

8 **a** inverse, one-one
b no inverse, many-one

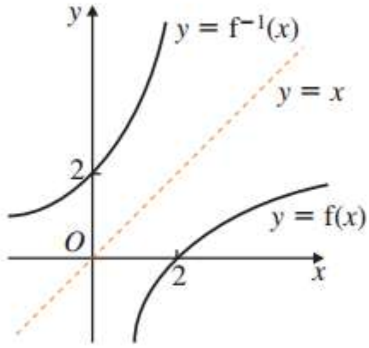
9 **a** $f^{-1}: x \mapsto \frac{x-5}{10}$, domain $x \in \mathbb{R}$, range $f^{-1}(x) \in \mathbb{R}$



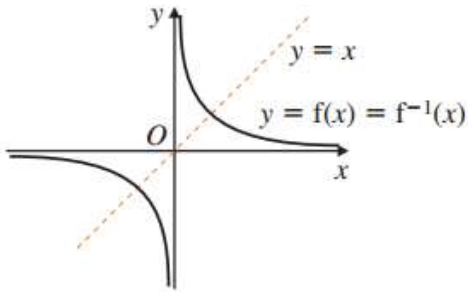
b $f^{-1}: x \mapsto \sqrt[3]{x}$, domain $x \in \mathbb{R}$, range $f^{-1}(x) \in \mathbb{R}$



c $f^{-1}: x \mapsto 10^x + 1$, domain $x \in \mathbb{R}$, range $f^{-1}(x) \in \mathbb{R}$, $f^{-1}(x) > 1$



d $f^{-1}: x \mapsto \frac{1}{x}$, domain $x \in \mathbb{R}$, $x \neq 0$, range $f^{-1}(x) \in \mathbb{R}$, $f^{-1}(x) \neq 0$



10 **a** $\frac{1}{4}$ **b** 6.5 **c** 7 or 9

11 **a** $f(x) \geqslant -1$

b $\sqrt{x+1} - 1$, domain $x \in \mathbb{R}$, $x \geqslant -1$, range $f^{-1}(x) \in \mathbb{R}$, $f^{-1}(x) \geqslant -1$

c -1 or 0

7.

SKILLS CHECK **1A EXTRA: Answers**

1 **a** $\frac{x+1}{2}$ **b** $\frac{x+3}{x-1}$ **c** $\frac{2x-3}{x+7}$ **d** $\frac{x-5}{2x}$

2 **a** $\frac{1}{2}$ **b** $\frac{x}{2}$ **c** $\frac{y}{y-1}$ **d** $\frac{x}{4x+3}$

3 **a** $\frac{2x+17}{(x-5)(2x-1)}$ **b** $\frac{4t-1}{2t+1}$ **c** $\frac{3}{(x-4)}$ **d** $\frac{x^2+5x+12}{(x-2)(x+2)(x+4)}$

4 $2x^2 + x - 1 + \frac{8}{2x-1}$

5 **a** $\frac{3}{(1-x)(2x-5)}$ **b** $x = -\frac{1}{2}$ or 4

6 Quotient is $3x^2 - 2x - 6$, remainder is $5x + 11$.

3A EXTRA: Answers

1 **a** $-\frac{2}{3}, 1, -\frac{5}{4}$ **b** $\pm 4, \pm 5$ **c** $\frac{1}{3}, -\frac{1}{2}, \frac{2}{3}$ **d** $\frac{5}{4}, -3$

2 **b** $(x+1)(x-2)(x-3)$ **c** $-1, 2, 3$

3 $(x-2)(x+2)^2$

6 $3, 1+\sqrt{5}, 1-\sqrt{5}$

7 $(x+3)^2(x-1)$

10 **a** $f(2) = 8 - 14 + 6 = 0 \Rightarrow (x-2)$ is a factor

b $f(-3) = -27 + 21 + 6 = 0 \Rightarrow (x+3)$ is a factor

c $f(x) = (x-2)(x+3)(x-1)$

d $(0, 6)$

e $(-3, 0), (1, 0), (2, 0)$

f