SJSF Sleaford Joint Sixth Form

A-level Further Mathematics Induction Booklet

Contents

Further Mathematics - Core

Further Mathematics consists of four papers:- two Core papers and two optional papers from a choice of further statistics, further mechanics and decision maths.

Further Mathematics builds on many areas of A Level Mathematics but also introduces new areas.

The areas include:
Proof, Complex numbers, Matrices, Further algebra and functions, Further calculus, Further vectors, Polar coordinates, Hyperbolic functions, Differential equations

This booklet introduces an area of Mathematics which is probably new to you. Enjoy!

Matrices

Section 1: Introduction to matrices

- A **matrix** is a rectangular array of numbers arranged in rows and columns, e.g. $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 1 & 7 \end{pmatrix}$.
- A matrix is usually denoted by a bold capital letter, e.g. $\mathbf{A}$ (or A̲ when hand-written).
- The matrix $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 1 & 7 \end{pmatrix}$ has 2 rows and 3 columns and so is described as a 2×3 matrix. The **order** of this particular matrix is said to be 2×3.

In general a $m \times n$ matrix will have m rows and n columns.

1.1 Special matrices

- A matrix with a single column, e.g. $\begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$, is called a **column vector**. Likewise a matrix with just one row, e.g. (1 -4), is called a **row vector**. We usually label column vectors and row vectors by lower case bold letters, e.g.

$$\mathbf{a} = \begin{pmatrix} 2 & 1 & -9 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -6 \\ 3 \end{pmatrix}.$$

- If a matrix has an equal number of rows and columns, it is called a **square matrix**, e.g. $\begin{pmatrix} 0 & 7 \\ -2 & 4 \end{pmatrix}$ is an example of a 2×2 square matrix.
- A **zero matrix** (or **null matrix**) has all its entries as 0. For example, a 4×2 zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- A **diagonal matrix** has zeros everywhere except on the **leading diagonal**, e.g.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ or } \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

- An **identity matrix** is a diagonal matrix that has 1's on the leading diagonal, e.g. the 2×2 identity matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the 3×3 identity matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

1.2 General notation

A general $m \times n$ matrix can be written as $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$.

The **entry** (or **element**) in the *i*th row and *j*th column is denoted a_{ij}.

1.3 Addition and subtraction of matrices

Two matrices can only be added or subtracted if they have the same order (i.e they have the same number of rows and columns).

To add (or subtract) two matrices, we add (or subtract) the corresponding entries.

Example:

If $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & -3 \\ 2 & -4 & 3 & 0 \\ 6 & 0 & -1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -3 & 7 & 6 \\ -4 & -5 & 0 & -2 \\ 2 & 4 & -3 & 5 \end{pmatrix}$ then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 5 & -1 & 8 & 3 \\ -2 & -9 & 3 & -2 \\ 8 & 4 & -4 & 10 \end{pmatrix} \text{ and}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & 5 & -6 & -9 \\ 6 & 1 & 3 & 2 \\ 4 & -4 & 2 & 0 \end{pmatrix}.$$

Questions

Find A + B and A-B for the following matrices

a) $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}$

b) $A = \begin{pmatrix} -4 & 5 \\ 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & -3 \\ 0 & 11 \end{pmatrix}$

c) $A = \begin{pmatrix} -11 & 13 \\ 28 & 43 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & 18 \\ 22 & -3 \end{pmatrix}$

d) $A = \begin{pmatrix} -3 & 7 & 6 \\ -5 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & 4 & 3 \\ -2 & -3 & 1 \end{pmatrix}$

1.4 Multiplication by a scalar

The matrix $k\mathbf{A}$, where k is a scalar (i.e. a single number), is obtained by multiplying all of the entries in matrix **A** by k.

Example:

If $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 0 & 1 \\ 4 & 2 \end{pmatrix}$ then $2\mathbf{A} = \begin{pmatrix} -4 & 6 \\ 0 & 2 \\ 8 & 4 \end{pmatrix}$ and $-3\mathbf{A} = \begin{pmatrix} 6 & -9 \\ 0 & -3 \\ -12 & -6 \end{pmatrix}$.

1.5 Multiplication of matrices

For two matrices **A** and **B**, the product **AB** is only defined if the number of columns of **A** is the same as the number of rows in **B**.
To calculate **AB** we multiply each row of A by each column of B.
We can illustrate this process by an example:

Suppose **A** = $\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 5 \end{pmatrix}$ and **B** = $\begin{pmatrix} 5 & 3 & -2 \\ 4 & 1 & 2 \end{pmatrix}$

then

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 3 & -2 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times4 & 1\times3+2\times1 & 1\times(-2)+2\times2 \\ 3\times5+(-1)\times4 & 3\times3+(-1)\times1 & 3\times(-2)+(-1)\times2 \\ 2\times5+5\times4 & 2\times3+5\times1 & 2\times(-2)+5\times2 \end{pmatrix} = \begin{pmatrix} 13 & 5 & 2 \\ 11 & 8 & -8 \\ 30 & 11 & 6 \end{pmatrix}.$$

General result
If **A** is of size $m \times n$ and **B** is of size $n \times p$ then **AB** is of size $m \times p$,
i.e. $(\text{m} \times \cancel{\text{n}}) \times (\cancel{\text{n}} \times \text{p}) = (\text{m} \times \text{p})$

Example:

If **A** = $\begin{pmatrix} 1 & 3 & 2 \\ 4 & -2 & 5 \\ -1 & 2 & -3 \end{pmatrix}$, **B** = $\begin{pmatrix} 2 & 0 \\ -1 & 5 \\ 2 & -3 \end{pmatrix}$ and **C** = $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ then

- **AB** = $\begin{pmatrix} 1 & 3 & 2 \\ 4 & -2 & 5 \\ -1 & 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 5 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2-3+4 & 0+15-6 \\ 8+2+10 & 0-10-15 \\ -2-2-6 & 0+10+9 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 20 & -25 \\ -10 & 19 \end{pmatrix}$.
- **AC** = $\begin{pmatrix} 1 & 3 & 2 \\ 4 & -2 & 5 \\ -1 & 2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4+3+4 \\ 16-2+10 \\ -4+2-6 \end{pmatrix} = \begin{pmatrix} 11 \\ 24 \\ -8 \end{pmatrix}$.
- **BC** cannot be calculated as the number of columns in **B** does not equal the number of rows in **C**.

Note:
AB will not in general equal **BA**. If **AB** exists, then **BA** does not necessarily exist. This means that matrix multiplication is not *commutative*.

Note 2:
Pre- or post-multiplying a matrix by the identity matrix (of appropriate size) will not alter it.

E.g. $\begin{pmatrix} 4 & -1 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 2 & 5 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}$.

- So we have the result that for a general matrix **A**,

IA = AI = A

1.6 The transpose of a matrix

To find the *transpose* of a matrix **A** you swap the rows and columns over.

Example:

If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \\ -2 & 4 \end{pmatrix}$ then the transpose of **A**, written $\mathbf{A}^{\mathrm{T}}$, is $\mathbf{A}^{T} = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 4 \end{pmatrix}$.

Example 2:

If $\mathbf{B} = \begin{pmatrix} 4 & 4 & 2 \\ 5 & 9 & 6 \\ -7 & 3 & 1 \end{pmatrix}$, then $\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 4 & 5 & -7 \\ 4 & 9 & 3 \\ 2 & 6 & 1 \end{pmatrix}$.

Questions

$$A = \begin{pmatrix} 1 & 5 \\ -1 & 2 \\ 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 8 \\ 4 & 1 & -2 \end{pmatrix}$$

Find

1. **2A**
2. **-3B**
3. $\mathbf{B}^{\mathrm{T}}$

4. Find AB for each of the following

a) $A = \begin{pmatrix} 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

b) $A = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -7 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 8 \\ 0 & 11 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$

d) $A = \begin{pmatrix} 4 & 8 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ 5 & -2 \end{pmatrix}$

e) $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 2 \\ 4 & 3 \end{pmatrix}$

f) $A = \begin{pmatrix} 2 & 1 \\ 0 & 4 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 \\ 4 & 1 \end{pmatrix}$

Section 2: Determinants and inverses of 2 by 2 matrices

2.1 Determinant of a 2 by 2 matrix

The determinant of the 2 × 2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as the number

det(**A**) = ad – bc.

Example 1:

If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$, then det(**A**) = (2×5) – (3×1) = 10 – 3 = 7.

Example 2:

If $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$, then det(**B**) = (2×3) – (-1×4) = 6 + 4 = 10.

2.2 Singular matrices

A matrix is called singular if its determinant is 0.

Example:

$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix}$ is a singular matrix since det(**A**) = 12 – 12 = 0.

Worked examination style question

Find the restrictions needed on the value of k if the matrix $\mathbf{A} = \begin{pmatrix} k & 3 \\ 4 & k-4 \end{pmatrix}$ is to be non-singular.

Solution: **A** is non-singular if det(**A**) ≠ 0.

But det(**A**) = $k(k-4) - 12 = k^2 - 4k - 12 = (k+2)(k-6)$.

So we require:

$$(k + 2)(k - 6) \neq 0$$

i.e. $k \neq -2$ and $k \neq 6$.

2.3 The inverse of a 2 × 2 matrix

Suppose that two (square) matrices **A** and **B** are such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

A is then called the *inverse* of **B** (and **B** is the inverse of **A**). We write $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$.

Example:

Note that $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So the inverse of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ and the inverse of $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ is $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

General result

There are two steps in order to find the inverse of a general matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Step 1: Swap the two elements on the leading diagonal and change the sign of the other two entries, giving the matrix:

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Step 2: Divide each of the entries in this new matrix by the *determinant* of **A**.

So we have that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Note: When the determinant of a matrix is zero, the inverse matrix is does not exist. So we can only find inverses for non-singular matrices.

Example:

Find the inverse of $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 2 & 2 \end{pmatrix}$.

Solution:

The determinant of **B** is det **B** = (4 × 2) – (–1 × 2) = 10.

So the inverse of **B** is

$$\mathbf{B}^{-1} = \frac{1}{10}\begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}.$$

Note: As a check we could work out $\mathbf{BB}^{-1}$ and make sure this product gives the identity matrix.

Example 2:

Find the inverse of **C** = $\begin{pmatrix} 1 & 0 \\ 6 & -5 \end{pmatrix}$.

As det **C** = –5, the inverse matrix is

$$\mathbf{C}^{-1} = \frac{1}{-5}\begin{pmatrix} -5 & 0 \\ -6 & 1 \end{pmatrix} = \frac{-1}{5}\begin{pmatrix} -5 & 0 \\ -6 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 6 & -1 \end{pmatrix}$$

Questions

Find A^{-1} for each of the following

a) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

b) $A = \begin{pmatrix} -3 & -2 \\ 3 & 3 \end{pmatrix}$

Section 3: Solving simultaneous equations in two variables using matrices

3.1 Introduction to solving equations using matrices

The equations

$$ax + by = \alpha$$
$$cx + dy = \beta$$

can be represented in matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

From the work in the previous section, we can find the solution vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by multiplying by the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Note: A unique solution exists provided that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular.

Example:
Using a matrix method, solve:

$$3x - y = 5$$
$$-4x + 2y = -9.$$

Solution:

The equations can be written in matrix form: $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \end{pmatrix}$.

We find the inverse of the matrix: $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$.

To find the solutions, we multiply through by this inverse matrix:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3.5 \end{pmatrix}.$$

So the solution to the equations are: $x = 0.5$ and $y = -3.5$.

Examination style question:
Using a matrix method, solve the equations:

$$2x + 5y = 2$$
$$-3x - 8y = -2$$

.,