Carre’s Grammar School A level Maths Induction Booklet

Bridging the gap from GCSE to A level

POR DYSSERVER

Introduction

At A level it is assumed that you are comfortable with the majority of the topics at GCSE grade A and A*.

This booklet is designed to give you a concise set of notes on these topics and worked examples to be able to answer these types of questions.

We expect that you will have taken the time to revise and recap on these topics so that you can access the A level course effectively. To ensure you are on the correct programme of study we will be testing you on these topics in your first week of A level.

Lesson 1 - Surds

A surd is a number which cannot be written exactly without square roots

Eg. $\sqrt{3}$ is a surd because it can't be written exactly without a root, $\sqrt{9}$ is not a surd because it can be simplified to 3.

There are three rules which will help you:

1) $\sqrt{ab} = \sqrt{a}\sqrt{b}$
2) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
3) $(\sqrt{a})^2 = \sqrt{a}\sqrt{a} = a$

Simplifying surds

Eg $\sqrt{12} = \sqrt{4x3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

Using rule (1) we can break the 12 into a product of two factors. We could have used other factors such as 2 and 6 but root 2 and root 6 cannot be simplified to an integer whereas by using 4 (a square number) and 3 we can then simplify the root 4 into 2.

TOP TIP: look to write the number in the square root as the product of biggest square and smallest prime

Eg $(2\sqrt{5} + 3\sqrt{6})^2 = 74 + 12\sqrt{30}$

Surds behave in the same manner as algebra, we can expand this by treating it as:

$(2\sqrt{5} + 3\sqrt{6})\ (2\sqrt{5} + 3\sqrt{6})$

You can then use any method you feel comfortable with to expand these brackets using FOIL and using rule (1) in reverse we get:

F: $2\sqrt{5} \times 2\sqrt{5} = 4\sqrt{25} = 4 \times 5 = 20$

O: $2\sqrt{5} \times 3\sqrt{6} = 6\sqrt{30}$

I: $3\sqrt{6} \times 2\sqrt{5} = 6\sqrt{30}$

L: $3\sqrt{6} \times 3\sqrt{6} = 9\sqrt{36} = 9 \times 6 = 54$

So we now have:

$20 + 6\sqrt{30} + 6\sqrt{30} + 54$

Giving us: $74 + 12\sqrt{30}$

Rationalising the denominator

When dealing with surds in fractions we never leave a surd as the denominator. We create an equivalent fraction where the surd no longer appears on the bottom, this is called rationalising the denominator. Thinking about numeric fractions:

e.g. $\frac{2}{5}$

We can create an equivalent fraction by multiplying the numerator and denominator by the same number OR effectively multiplying by a fraction with the same numerator and denominator:

$\frac{2}{5} \times \frac{3}{3} = \frac{6}{15}$

We can apply the same principle along with surd rule (3)

e.g. $\frac{9}{\sqrt{3}}$

The surd appears in the denominator so we need to multiply it by a fraction containing a surd that will remove the root 3 from the denominator

$\frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

By multiplying these fractions we now get:

$\frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{9\sqrt{3}}{3}$

This fraction can then be simplified by cancelling out a factor of 3 from numerator and denominator to give:

$3\sqrt{3}$

Lesson 2 – Indices

Raising a number or an expression to a power / index / exponent represents repeated multiplication of the number or expression

Eg 2^3 represents $2 \times 2 \times 2$ (2 is the base number and 3 is the power / order / exponent)

Laws of indices

Increasing the power by one represents multiplying by the base number one more time. Expressions with the same base can therefore be multiplied as follows

$$4^2 \times 4^3 = (4 \times 4) \times (4 \times 4 \times 4) = 4^5$$

Giving $\boldsymbol{x^a \times x^b = x^{a+b}}$

Similarly, for division $4^5 \div 4^3 = \frac{4\times4\times4\times4\times4}{4\times4\times4} = 4^{5-3} = 4^2$

And in general $\boldsymbol{x^a \div x^b = x^{a-b} = \frac{x^a}{x^b}}$

Also, expressions raised to a power can be raised to a further power as follows

$$(4^2)^3 = 4^2 \times 4^2 \times 4^2 = 4^6$$

So in general $\boldsymbol{(x^a)^b = x^{ab} = (x^b)^a}$

Negative indices

Decreasing the power by one represents dividing by the base number.

Eg $5^2 = 5 \times 5 = 25$

$5^1 = 5$ (power decreased by 1, value divided by 5)

And therefore it follows that

$5^0 = 5 \div 5 = 1$ (power decreased by 1, value divided by 5)

And that

$5^{-1} = 1 \div 5 = \frac{1}{5}$

$5^{-2} = \frac{1}{5} \div 5 = \frac{1}{25} \; or \; \frac{1}{5^2}$ thus

$$\boldsymbol{a^{-x} = \frac{1}{a^x}}$$

Fractional Indices

What is the value of $9^{1/2}$?

Since, by the first rule of indices $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^1$

Then $9^{1/2}$ represents the number which, when multiplied by itself, gives 9 – ie the square root of 9.

Thus $$\boldsymbol{x^{\frac{1}{a}} = \sqrt[a]{x}}$$

Combined Indices

An index can be split into its component parts using the third law of indices.

For example, since $\frac{-3}{2} = \frac{1}{2} \times 3 \times -1$

$$9^{\frac{-3}{2}} = (((9^{\frac{1}{2}})^3)^{-1})$$

And working outwards from the innermost brackets can be written as

$$= ((3^3)^{-1})$$

$$= 27^{-1}$$

$$= \frac{1}{27}$$

Thus $$\boldsymbol{x^{\frac{a}{b}} = \sqrt[b]{(x^a)} = (\sqrt[b]{x})^a \text{ and } x^{\frac{-a}{b}} = \frac{1}{\sqrt[b]{(x^a)}} = \frac{1}{(\sqrt[b]{x})^a}}$$

Expressions with coefficients

You know that $3x^2$ represents $3 \times x^2$. The only part of the expression raised to the power of 2 is the base x and therefore the index 2 has no impact on the coefficient 3.

Similarly, $\frac{3}{x^2}$ can be written as $3 \times \frac{1}{x^2}$ or $3x^{-2}$

This logic continues $\frac{5}{2x^4}$ represents $\frac{5}{2} \times \frac{1}{x^4}$

Which can be written as $\frac{5}{2}x^{-4}$

Thus $$\boldsymbol{\frac{px^a}{q} = \frac{p}{q} \times x^a \text{ and } \frac{p}{qx^a} = \frac{p}{q} \times \frac{1}{x^a} = \frac{p}{q}x^{-a}}$$

Lesson 3 - Algebraic skills

It is assumed that at A level you have good basic algebraic skills, these include algebraic manipulation for solving equations and similarly you should be able to rearrange equations competently including questions where the variable appears multiple times.

We can use algebra to change the subject of a formula. Rearranging a formula is similar to solving an equation – we must do the same to both sides in order to keep the equation balanced.

Example 1: Make x the subject of the formula $y = 4x + 3$.

Solution: $y = 4x + 3$

Subtract 3 from both sides: $y - 3 = 4x$

Divide both sides by 4; $\frac{y-3}{4} = x$

So $x = \frac{y-3}{4}$ is the same equation but with x the subject.

Example 2: Make x the subject of $y = 2 - 5x$

Solution: Notice that in this formula the x term is negative.

$y = 2 - 5x$

Add $5x$ to both sides $y + 5x = 2$ (the x term is now positive)

Subtract y from both sides $5x = 2 - y$

Divide both sides by 5 $x = \frac{2-y}{5}$

Example 3: The formula $C = \frac{5(F-32)}{9}$ is used to convert between ° Fahrenheit and ° Celsius.

We can rearrange to make *F* the subject.

$$C = \frac{5(F-32)}{9}$$

Multiply by 9 $\quad 9C = 5(F-32)$ (this removes the fraction)

Expand the brackets $\quad 9C = 5F - 160$

Add 160 to both sides $\quad 9C + 160 = 5F$

Divide both sides by 5 $\quad \frac{9C+160}{5} = F$

Therefore the required rearrangement is $F = \frac{9C+160}{5}$.

Example 4: Make *x* the subject of $x^2 + y^2 = w^2$

Solution: $\quad x^2 + y^2 = w^2$

Subtract y^2 from both sides: $\quad x^2 = w^2 - y^2$ (this isolates the term involving *x*)

Square root both sides: $\quad x = \pm\sqrt{w^2 - y^2}$

Remember that you can have a positive or a negative square root. We cannot simplify the answer any more.

Example 5: Make *a* the subject of the formula $t = \frac{1}{4}\sqrt{\frac{5a}{h}}$

Solution: $t = \frac{1}{4}\sqrt{\frac{5a}{h}}$

Multiply by 4 $4t = \sqrt{\frac{5a}{h}}$

Square both sides $16t^2 = \frac{5a}{h}$

Multiply by *h*: $16t^2 h = 5a$

Divide by 5: $\frac{16t^2 h}{5} = a$

Sometimes the variable that we wish to make the subject occurs in more than one place in the formula. In these questions, we collect the terms involving this variable on one side of the equation, and we put the other terms on the opposite side.

Example 6: Make *t* the subject of the formula $a - xt = b + yt$

Solution: $a - xt = b + yt$

Start by collecting all the t terms on the right hand side:

Add *xt* to both sides: $a = b + yt + xt$

Now put the terms without a *t* on the left hand side:

Subtract *b* from both sides: $a - b = yt + xt$

Factorise the RHS: $a - b = t(y + x)$

Divide by (*y* + *x*): $\frac{a-b}{y+x} = t$

So the required equation is $t = \frac{a-b}{y+x}$

Example 7: Make *W* the subject of the formula $T - W = \frac{Wa}{2b}$

Solution: This formula is complicated by the fractional term. We begin by removing the fraction:

Multiply by 2*b*: $2bT - 2bW = Wa$

Add 2*bW* to both sides: $2bT = Wa + 2bW$ (this collects the W's together)

Factorise the RHS: $2bT = W(a + 2b)$

Divide both sides by *a* + 2*b*: $W = \frac{2bT}{a + 2b}$

Lesson 4 - Solving Quadratic Equations

A quadratic equation is any equation where x^2 is the highest power of x.

There are two key forms that you are expected to be able to write a quadratic as, these are:

●Factorised form

●Completed the square form

Factorising

No Coefficient of x^2:
If there is no coefficient of x^2 then you can use a short method to factorise.
Given a quadratic in the form: $x^2 + \mathbf{b}x + \mathbf{c}$
Then when you factorise into double brackets you will have to find a pair of numbers that multiply to make **c** and add to make **b**. These numbers (lets call them **d** and **e**) then go in double brackets as follows:
$(x + \mathbf{d}) (x + \mathbf{e})$
e.g. $x^2 + 5x – 24$
In this example **b** = +5 and **c =** -24
-24 = -3 x 8
+5 = -3 + 8
Therefore our factorised form is:
$(x – 3) (x + 8)$

Coefficient of x^2:
If there is a coefficient of x^2 the method above will not work.
Given a quadratic in the form **a**x^2 + **b**x + **c**
We are still looking for a pair of numbers but they will not go directly into our brackets.
b is still the number that the pair must add to make
a x **c** however is the number that they multiply to make.
e.g. $15x^2 + 19x + 6$
in this example **a** = 15, **b** = 19, **c** = 6
Therefore we are looking for a pair of numbers that:
multiply to make 15 x 6 = 90
add to make 19

The pair of numbers therefore are 9 and 10

These numbers then could be put into a grid to help us visualise what is going on:

$15x^2$	+9x
+10x	+6

If we then take out the highest common factor of each row and each column we get:

	5x	+3
3x	$15x^2$	+9x
+2	+10x	+6

This then gives us our double brackets of:
(5x + 3) (3x + 2)

Completing the square

Another vital skill is completing the square. This is where we put a quadratic in the form:

$(x + d)^2 + e$

To find the value of **d** from the quadratic we take the value of **b** and divide it by 2. To find the value of **e** we take the value of **c** from the quadratic and subtract $\mathbf{d^2}$ from it (N.B. We always subtract $\mathbf{d^2}$ even if **d** is a negative number)

e.g. $x^2 - 6x+8$
Dividing the coefficient of x (-6 in this case) by 2 gives -3
Subtracting $(-3)^2$ from 8 gives -1
Therefore the answer is $(x-3)^2 -1$

Solving Quadratics

To be able to solve a quadratic equation there is one key point before you begin, your equation **MUST EQUAL ZERO**, if it does not, you must rearrange it so that it does.

e.g. $x^2 -2x – 99 = 0$

Factorising this using the aforementioned method gives:

$(x – 11) (x + 9) = 0$

Since this means bracket 1 multiplied by bracket 2 is equal to zero, one of the brackets must equal zero.

Therefore:

$x – 11 = 0$ or $x + 9 = 0$

Leading to:

$x = 11$ or $x = -9$

We could also have solved this equation using **Completing the Square**:

Completing the square for . $x^2 -2x – 99 = 0$ gives:

$(x - 1)^2 - 100 = 0$

We can then solve this as we would a linear equation:

$(x - 1)^2 = 100$

$x - 1 = (+/-\sqrt{100})$ (Whenever we root we get a positive and negative answer)

$x = (+/-\sqrt{100}) + 1$

$x = 10 + 1$ or $x = -10 +1$

$x = 11$ or $x = 9$

Lesson 5 - Algebraic Fractions

Simplifying

Consider simple numeric fractions, and how you would cancel them down, you would divide the numerator and denominator by the same value. You can show this in the following way:

e.g. 1) $\frac{3}{9} = \frac{1 \times 3}{3 \times 3} = \frac{1}{3}$

We have expressed the numerator and denominator as a **PRODUCT** of two factors, one being common. We can then cancel the common factor from the numerator and denominator to leave the simplified fraction.

Now consider the common mistake shown in e.g. 2 :

e.g. 2) $\frac{3}{9} = \frac{2+1}{2+7} = \frac{1}{7}$

Clearly 3/9 does not equal 1/7. Yet when we move on to look at algebraic fractions people will make this mistake. You have expressed the numerator and denominator here as a **SUM** not a **PRODUCT.** Therefore you cannot cancel it from the numerator and denominator.

Now let's begin looking at algebra:

e.g. 3) $\frac{3y^5}{9y^2} = \frac{3 \times y \times y \times y \times y \times y}{3 \times 3 \times y \times y} = \frac{y^3}{3}$

You could have used laws of indices here to have simplified it quickly rather than writing the entire calculation out.

Now let us begin to consider questions where x occurs more than once, or where we have a quadratic.

e.g. 4) $\frac{y^2 + y}{y^2 + 3y + 2}$

The temptation is to cancel the y^2 from the numerator and denominator as it appears in both, however this goes back to the point made in e.g. 2. It is expressed in both numerator and denominator as a **SUM** not as a product of **FACTORS.** Which leads us on to the correct method- to get it into a form where we have factors i.e. *factorise.*

e.g. 4) $\dfrac{y^2 + y}{y^2 + 3y + 2} = \dfrac{y(y+1)}{(y+1)(y+2)} = \dfrac{y}{(y+2)}$

Adding / Subtracting Fractions

To add fractions, you require a common denominator. Unless it is obvious, an easy way to ensure a common denominator is to multiply the denominators. You then have to do to the numerator what you have done to the denominator. This is shown below:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

<u>Numeric example</u>

$$\frac{3}{5} + \frac{2}{7} = \frac{3 \times 7 + 5 \times 2}{5 \times 7} = \frac{21 + 10}{35} = \frac{31}{35}$$

This particular example does not simplify but you should always check, especially when working with algebraic fractions.

<u>Algebraic Example</u>

$$\frac{2b}{a} + \frac{a-b}{b} = \frac{2b(b) + a(a-b)}{ab} = \frac{2b^2 + a^2 - ab}{ab}$$

There are no common factors in the numerator therefore it cannot be factorised. Therefore this fraction does not simplify. *You may be tempted to cancel the ab from the numerator and denominator but it is a **SUM** not a **PRODUCT!!***

Multiplying Fractions

As with numeric fractions, you simply multiply numerators, multiply denominators:

$$\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}$$

Now let us consider some algebraic fractions:

$$\frac{3a^2b^3}{4a^2b} \times \frac{2a^3b}{5ab^3} = \frac{6a^5b^4}{20a^3b^4} = \frac{3a^2}{10}$$

This question is completed using your index rules for multiplication and division. Just remember that where you have an unknown with no power stated, the power is always 1.

Also, note that after the initial calculation is completed, you still have to cancel down where appropriate.

TOP TIP: This only applies when multiplying, you can cancel from the diagonals before you even do the calculation. E.g. consider the numerator of the second fraction and the denominator of the first fraction. You could have cancelled the 2 and 4 down to 1 and 2 etc. before the rest of the calculation was completed.

Dividing fractions

To divide fractions, as with numeric examples you flip the second fraction and then change the divide symbol to a multiplication:

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{bc}$$

Now let us consider an algebraic fraction:

$$\frac{2x^2y^3z^7}{9x^3yz^9} \div \frac{8x^3y^5z}{3x^4z^7} = \frac{2x^2y^3z^7}{9x^3yz^9} \times \frac{3x^4z^7}{8x^3y^5z} = \frac{6x^6y^3z^{14}}{72x^6y^6z^{10}} = \frac{z^4}{12y^3}.$$

Again, this uses the indices work you have already completed. Key point, just remember to cancel/simplify at the end. You can simplify here as both numerator and denominator are **PRODUCTS** with common factors.

Solving Equations

If we have as algebraic equation that involves fractions, the easiest way to deal with these is to get rid of the fractions. To do this you will need to multiply through by each of the denominators. This is illustrated below:

e.g. $\frac{3y + 2}{4} = y - 1$ To get rid of the 4 as the denominator, multiply *EVERYTHING* by 4

$3y + 2 = 4(y - 1)$ Note that it is the entirety of the right hand side that is multuiplied. Now you can just solve as any linear equation.

If we have more than one part, we have to be careful:

e.g. 2 $\frac{4y + 6}{3} = \frac{2y + 3}{2}$ To begin, we need to multiply through by each denominator

$4y + 6 = \frac{3(2y + 3)}{2}$ And then the other denominator

$2(4y + 6) = 3(2y + 3)$ And again, we now have a simple linear equation.

A shortcut can be to simply multiply each term in the equation by any denominator not attached to it. Using e.g. 2, the LHS is multiplied by the 2 and the RHS is multiplied by the 3. This method can save a lot of time in questions such as:

e.g. 3 $\frac{1}{x - 3} - \frac{2}{x - 1} = \frac{1}{3}$ using the above rule gives:

$3(x - 1)(1) - (x - 3)(3)(2) = (x - 3)(x - 1)(1)$

Then expand the brackets, you will have a quadratic.

$3x - 3 - (6x - 18) = x^2 - 4x + 3$

Be careful with signs here, it is the entire bracket subtracted.

$-3x + 15 = x^2 - 4x + 3$ Now make it equal zero and solve.

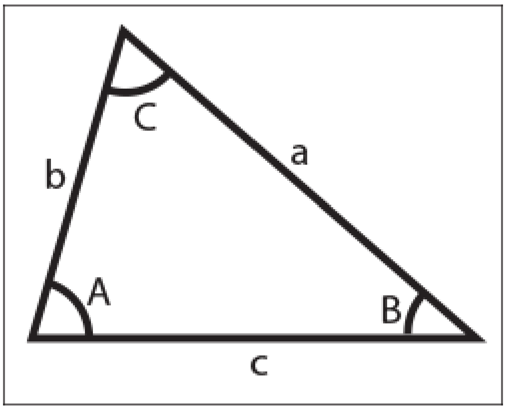
$0 = x^2 - x - 12$ Factorise/ complete the square/ formula

$0 = (x - 4)(x + 3)$ Therefore $x = 4$ or $x = -3$

Lesson 6 – Trigonometry

It is assumed that you are capable of tackling questions involving non right angled triangles and are able to calculate missing side lengths or angles.

For any non right angled triangle we label it in the following way:



It does not matter which side/ angle you label as a, b or c as long as opposite sides and angles have the same letter. Uppercase letters refer to angles, lowercase letters refer to sides. It is always useful to label the triangle with these letters.

Sine Rule

We use Sine rule when we are given a value of an angle and a side that are opposite one another i.e. a and A, b and B, c and C

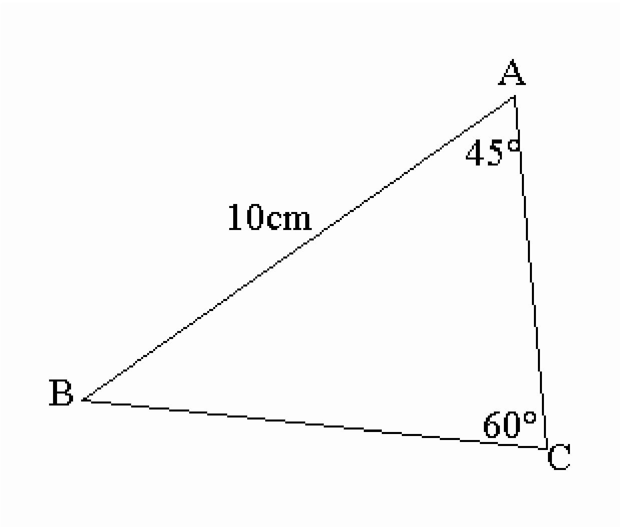
Sine Rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \Leftrightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The version on the left is easiest for finding angles; the version on the right is easiest for finding side lengths. You only need to use two of the three fractions to calculate either a side or an angle.

Once you have identified that it is Sine rule, you simply substitute the values you have labelled in.

e.g. Finding a side length using Sine Rule:



You are given angles A and C and side c, so use

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$ State the formula

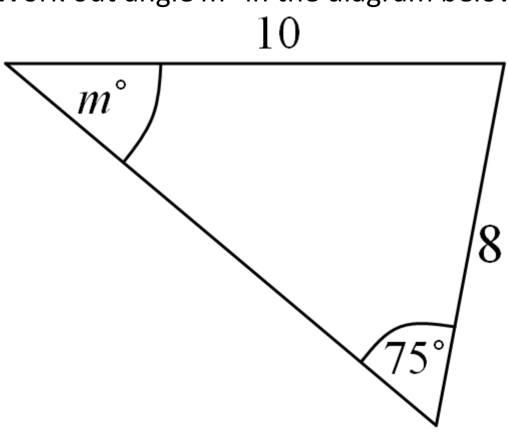
$$\frac{a}{\sin 45} = \frac{10}{\sin 60}$$ Substitute the values you know

$$a = \frac{10 \times \sin 45}{\sin 60}$$ Rearrange the equation

$$a = 8 \cdot 16cm$$ Solve to find the solution

e.g. Finding an angle using Sine Rule:

Work out angle *m*° in the diagram below:



Step 1 Start by writing out the Sine Rule formula for finding angles:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

Step 2 Fill in the values you know, and the unknown angle:

$$\frac{\sin(m°)}{8} = \frac{\sin(75°)}{10}$$

Remember that each fraction in the Sine Rule formula should contain a side and its opposite angle.

Step 3 Solve the resulting equation to find the sine of the unknown angle:

$$\frac{\sin(m°)}{8} = \frac{\sin(75°)}{10}$$ *(multiply by 8 on both sides)*

$$\sin(m°) = \frac{\sin(75°)}{10} \times 8$$

$\sin(m°) = 0.773$ (3 significant figures)

Step 4 Use the inverse-sine function ($\sin^{-1}$) to find the angle:

$m° = \sin^{-1}(0.773) = 50.6°$ (3sf)

Cosine Rule

Cosine Rule is used when Sine Rule does not work i.e. we either have all 3 sides and are finding an angle or we have two sides with an angle in between and are finding the opposite side.

Using the same labelling of the triangle as set out earlier:

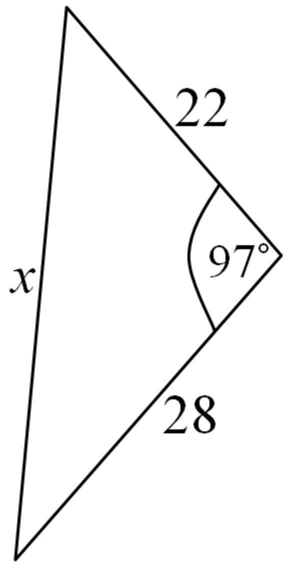
Cosine Rule:

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

Therefore it is useful to label whichever side or angle you are finding as **a** or **A**.

e.g. Finding a missing side length:

Work out the length of x in the diagram below:



Step 1 Start by writing out the Cosine Rule formula for finding sides:

$a^2 = b^2 + c^2 - 2bc\cos(A)$

Step 2 Fill in the values you know, and the unknown length:

$x^2 = 22^2 + 28^2 - 2\times22\times28\times\cos(97°)$

It doesn't matter which way around you put sides b and c – it will work both ways.

Step 3 Evaluate the right-hand-side and then square-root to find the length:

$x^2 = 22^2 + 28^2 - 2\times22\times28\times\cos(97°)$ *(evaluate the right hand side)*

$x^2 = 1418.143.....$ *(square-root both sides)*

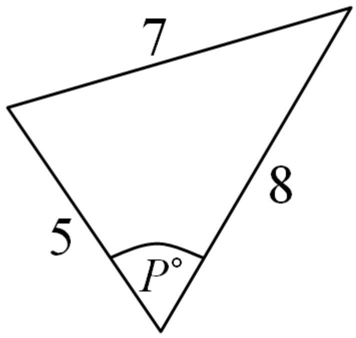
$x = 37.7$ (accurate to 3 significant figures)

As with the Sine Rule you should try and keep full accuracy until the end of your calculation to avoid errors.

e.g. Using Cosine Rule to find an angle:

The Cosine Rule formula can be rearranged to make Cos(A) the subject.

Work out angle $P°$ in the diagram below:



Step 1 Start by writing out the Cosine Rule formula for finding angles:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Step 2 Fill in the values you know, and the unknown length:

$$\cos(P°) = \frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}$$

Remember to make sure that the terms on top of the fraction are in the correct order.

Step 3 Evaluate the right-hand-side and then use inverse-cosine ($\cos^{-1}$) to find the angle:

$$\cos(P°) = \frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}$$ *(evaluate the right-hand side)*

$$\cos(P°) = 0.5$$ *(do the inverse-cosine of both sides)*

$$P° = \cos^{-1}(0.5) = 60° \text{ (3sf)}$$